Some Analytic Aspects of Vafa-Witten Twisted $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

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Geometry in four dimensions

Possibly the most prominent feature of four-dimensional geometry:

$$1 \longrightarrow \{\pm I\} \longrightarrow \mathrm{SO}(4) \longrightarrow \mathrm{SO}(3) \times \mathrm{SO}(3) \longrightarrow 1$$

If $\dim V = 4$, then

$$\begin{aligned} \Lambda^2 V &= \Lambda^+ V \oplus \Lambda^- V \\ 6 &= \;\;3 \;\;+\;\; 3 \end{aligned}$$

$$1 \longrightarrow \{\pm I\} \longrightarrow \mathrm{SO}(V) \longrightarrow \mathrm{SO}(\Lambda^+ V) \times \mathrm{SO}(\Lambda^- V) \longrightarrow 1$$

$\Lambda^+ V$ is an oriented three-dimensional vector space associated to any 4-D oriented Euclidean V.

The cross product

Let X be an oriented Riemannian four-manifold. For any $x \in X$, take $V = T_x^* X$.

- Choose an oriented orthonormal basis $\{e^0, e^1, e^2, e^3\}$.
- An oriented orthonormal basis for $\Lambda^+ T_x^* X$ is

$$\begin{aligned} \sigma^1 &= e^0 \wedge e^1 + e^2 \wedge e^3, \\ \sigma^2 &= e^0 \wedge e^2 + e^3 \wedge e^1, \\ \sigma^3 &= e^0 \wedge e^3 + e^1 \wedge e^2. \end{aligned}$$

- Define the cross product on $\Lambda^+ T_x^* X$ via $\{\sigma^i\}$ components, so $\sigma^1 \times \sigma^2 = \sigma^3$.

The de Rham complex

Inside the de Rham complex

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \xrightarrow{d} \Omega^4 \to 0$$

is the subcomplex

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^{2,+} \to 0.$$

$$b^0 \qquad b^1 \qquad b^+$$

Given a principal bundle P with connection A,

$$0 \to \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^+} \Omega^{2,+}(\mathfrak{g}_P) \to 0.$$

The double-composition is

$$d_A^+ \circ d_A = [F_A^+, \bullet].$$

This defines a complex when $F_A^+ = 0$.

Anti-self-dual equation

People often study the equation $F_A^+ = 0$.

It is called the anti-self-dual equation since $F_A^+ = 0 \iff F_A = F_A^-$.

Solutions arise as absolute minimizers of $\|F_A\|_{L^2}$.

If $\mathrm{g} \in \mathcal{G}_P$ is a gauge transformation, then

$$F^+_{\mathrm{g}(A)} = \mathrm{g} \cdot F_A^+ \cdot \mathrm{g}^{-1},$$

so $\mathcal{G}_P$ preserves solutions to $F_A^+ = 0$. The moduli space

$$\mathcal{M}_{\mathrm{ASD}} = \{[A] \in \mathcal{A}_P/\mathcal{G}_P | F_A^+ = 0\}$$

is finite-dimensional. Roughly speaking, it defines a homology class in $\mathcal{A}_P/\mathcal{G}_P$. This leads to Donaldson invariants.

Motivating question for studying Vafa-Witten

- What is the Euler characteristic of the ASD moduli space $\mathcal{M}_{\mathrm{ASD}}$?
  - Is this question meaningful?
    - $\mathcal{M}_{\mathrm{ASD}}$ can be singular
    - $\mathcal{M}_{\mathrm{ASD}}$ depends on the choice of a metric
    - $\mathcal{M}_{\mathrm{ASD}}$ has multiple possible compactifications

Comparison with Donaldson invariants

Write $\mathcal{M}_{\mathrm{ASD}}$ as a zero set:

$$\mathcal{M}_{\mathrm{ASD}} = \{[A] \in \mathcal{A}/\mathcal{G} \,|\, F_A^+ = 0\}.$$

If $\dim(\mathcal{M}_{\mathrm{ASD}}) = 0$, then the Donaldson invariant is a signed count $\#\mathcal{M}_{\mathrm{ASD}}$.

Analogy with polynomials:

- Let $M = \{x | x^2 - c = 0\}$. How many points are in m?
  - Signed count $\#M$ gives $+1 - 1 = 0$.
    - Generically well-defined on $\mathbb{R}$.
  - Unsigned count $\chi(M)$ gives $1 + 1 = 2$.
    - Generically well-defined on $\mathbb{C}$.

"Complexification" of configuration space

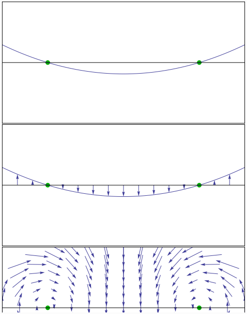
When $\dim(\mathcal{M}_{\mathrm{ASD}}) = 0$, we expect the signed count of the Donaldson invariant $\#\mathcal{M}_{\mathrm{ASD}}$ to typically differ from the unsigned count "$\chi(\mathcal{M}_{\mathrm{ASD}})$".

In analogy with complexification, we will "double" the degrees of freedom in our configuration space by adding extra fields. This leads to an augmented moduli space $\mathcal{M}_{\mathrm{VW}}$ with

$$\mathcal{M}_{\mathrm{ASD}} \subset \mathcal{M}_{\mathrm{VW}}.$$

Counting zeroes of a section

Consider a vector bundle $V \to X$ with a section $s : X \to V$.



Orienting the zeroes

Consider the horizontal/vertical components of the derivative along the zero section:

$$\begin{pmatrix} 0 & \bullet \\ ds & \bullet \end{pmatrix}$$

(The horizontal component vanishes identically along the zero section.) We can choose the $\bullet$.

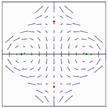
Choose the $\bullet$ as

$$\begin{pmatrix} 0 & ds^* \\ ds & 0 \end{pmatrix}$$

to achieve consistently-signed determinant along the zero-section.

Vanishing theorem

Ideally, our extended vector field will have no additional zeroes. This is the content of a "vanishing theorem."



$\ddot\smile$

If a vanishing theorem holds, the zeroes of our vector field agree with the zeroes of our section, and the signed zero count of our vector field equals the unsigned zero count of our section.

Euler characteristic of $\mathcal{M}_{\mathrm{ASD}}$

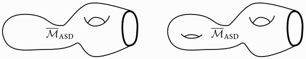
In this finite-dimensional analogy, $\mathcal{M}_{\mathrm{ASD}}$ is the zero-set of the section, and $\mathcal{M}_{\mathrm{VW}}$ is the zero-set of the vector field.

When a vanishing theorem holds, $\mathcal{M}_{\mathrm{ASD}} = \mathcal{M}_{\mathrm{VW}}$. In this case, we expect

$$\text{“}\#\mathcal{M}_{\mathrm{VW}}\text{”} = \text{“}\chi(\mathcal{M}_{\mathrm{ASD}}).\text{”}$$

What's wrong with $\chi(\mathcal{M}_{\mathrm{ASD}})$?

The Poincaré-Hopf index theorem only computes the Euler characteristic of a compact manifold. Since $\mathcal{M}_{\mathrm{ASD}}$ is non-compact, we need a compactification $\overline{\mathcal{M}}_{\mathrm{ASD}}$:

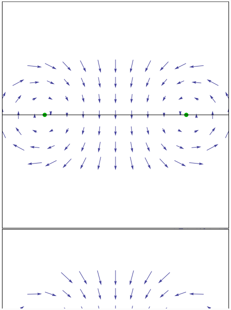


The invariant should be independent of the metric g, but different choices of g typically lead to *cobordant* compactified ASD moduli spaces $\overline{\mathcal{M}}_{\mathrm{ASD}}(g)$.

Euler characteristic is *not* invariant under cobordism! ($\mathbb{S}^2$)

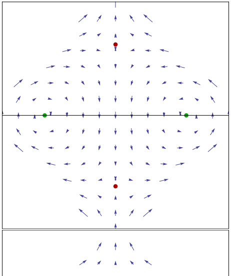
Behavior of families

Consider $0 = x^2 - c$ as c goes from positive to negative:



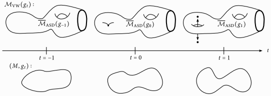
Alternative behavior

We could have extended the vector field differently.



What is the purpose of extra fields?

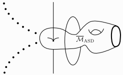
Consider a one-parameter family of metrics $\{g_t\}$ for $t \in \mathbb{R}$.



As the Euler characteristic of $\overline{\mathcal{M}}_{\mathrm{ASD}}$ changes, points in $\mathcal{M}_{\mathrm{VW}}$ should be created or destroyed to compensate.

Additional pathologies

The previous picture is a fantasy. The crux of this program is to understand how to deal with pathologies.



- Sequences of solutions in $\overline{\mathcal{M}}_{\mathrm{VW}}$ could have unbounded L^2 norms.
- Rays appear in $\overline{\mathcal{M}}_{\mathrm{VW}}$ at reducible points of $\overline{\mathcal{M}}_{\mathrm{ASD}}$.
- Despite having expected dimension zero, there are often manifolds of non-ASD solutions

Atiyah-Jeffrey supersymmetry

There is an Atiyah-Jeffrey style supersymmetric path integral expression for the Euler characteristic of $\mathcal{M}_{\mathrm{ASD}}$.

$$\text{“}\chi(\mathcal{M}_{\mathrm{ASD}})\text{”} = \text{“}\int e^{-L}.\text{”}$$

Vafa and Witten recognized Yamron's twist of $\mathcal{N} = 4$ supersymmetric Yang-Mills as such.

They were studying $\mathcal{N} = 4$ supersymmetry in the context of S-duality.

S-duality and geometric Langlands

In this context, S-duality roughly means that the generating function

$$\sum_k \text{“}\chi(\mathcal{M}_{\mathrm{ASD}}(k))\text{”} q^k$$

should be a modular form.

In several specific examples, they "computed" these generating functions and verified their modularity.

This Vafa-Witten theory is one of three twists of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. In 2006, Kapustin and Witten explored the relation of another such twist is to geometric Langlands. More recently, the Vafa-Witten twist has appeared in the work of Haydys and Witten on five-dimensional gauge theory.

Explicit example of S-duality

Consider the four-manifold $X = \mathrm{K3}$. The generating functions for $G = \mathrm{SU}(2)$ and $\hat{G} = \mathrm{SO}(3)$ are

$$Z_{\mathrm{SU}(2)}(q) = \tfrac{1}{2} q^{-2}\left(\tfrac{1}{4} + 0q + 30q^2 + 3200q^3 + \cdots \right.$$
$$\left. \cdots + \tfrac{10189790756178504975}{4} q^{16} + \cdots \right.$$

$$Z_{\mathrm{SO}(3)}(q) = q^{-2}\left(\tfrac{1}{4} + 0q^{1/2} + 0q + 2096128q^{3/2} + \right.$$
$$+ 50356230q^2 + 679145472q^{5/2} +$$
$$\left. \cdots + \tfrac{2137997440957227092282 4975}{4} q^{16} + \cdots \right.$$

Define $q^{1/2} = e^{i\pi\tau}$. In this case, S-duality is the "modular relation"

$$Z_{\mathrm{SU}(2)}(-1/\tau) = (2\tau)^{-12} Z_{\mathrm{SO}(3)}(\tau).$$

The equations

The equations

$$F_A^+ - \tfrac{1}{4}[B \times B] - \tfrac{1}{2}[C, B] = 0$$
$$d_A C + d_A^* B = 0$$

Let $P \to X^4$ be a principal bundle over an oriented Riemannian four-manifold. A *configuration* (C, A, B) consists of

- A section of the adjoint bundle $C \in \Omega^0(M; \mathfrak{g}_P)$
- A connection $A \in \mathcal{A}_P$
- An adjoint-valued self-dual two-form $B \in \Omega^{2,+}(M; \mathfrak{g}_P)$

The quadratic term

The equations

$$F_A^+ - \tfrac{1}{4}[B \times B] - \tfrac{1}{2}[C, B] = 0$$
$$d_A C + d_A^* B = 0$$

This quadratic term on $\mathfrak{g} \otimes \Lambda^{2,+}$ is the tensor product of the Lie bracket and the cross product.

Since $[,]$ is antisymmetric on $\mathfrak{g}$ and $\times$ is antisymmetric on $\Lambda^{2,+}$, their product $[B_1 \times B_2]$ is *symmetric*.

Only semi-definite

Note that $[B \times B]$ has a nontrivial kernel. Later we will see that this has dire consequences.

For example, if B has rank one

$$B = \chi \otimes \sigma^1,$$

then

$$[B \times B] = [\chi, \chi] \otimes (\sigma^1 \times \sigma^1) = 0 \otimes 0.$$

The quartic form $|[B \times B]|^2$ is only semi-definite.

The standard compactness strategy

- Use energy identities to establish a priori L_1^2 bounds
- These L_1^2 bounds imply weak compactness (Hodge theory for abelian case, Uhlenbeck/Sedlacek theory for non-abelian case)
- Elliptic regularity implies strong (Uhlenbeck) compactness

Summary

Using established analytic machinery, *a priori L_1^2 bounds imply compactness*

Examples: ASD, Seiberg-Witten, $\mathrm{PU}(2)$ monopoles

A priori estimates for SD/ASD equations

$$\mathcal{E}_{\mathrm{YM}}(A) := \tfrac{1}{2}\int_X |F_A|^2 .$$

By Uhlenbeck/Sedlacek theory, $\mathcal{E}_{\mathrm{YM}}(A)$ is analogous to $\|A\|^2_{L^2_1}$ after gauge fixing.

$$|F_A|^2 = |F_A^+|^2 + |F_A^-|^2 .$$

$$\kappa := \int_X \tfrac{1}{2}\left(|F_A^-|^2 - |F_A^+|^2\right),$$

proportional to a Chern number.

$$\mathcal{E}_{\mathrm{YM}}(A) = \int_X |F_A^-|^2 - \kappa = \int_X |F_A^+|^2 + \kappa.$$

Yang-Mills energy for ASD connections

$$\mathcal{E}_{\mathrm{YM}}(A) = \int_X |F_A^-|^2 - \kappa = \int_X |F_A^+|^2 + \kappa.$$

$$\begin{aligned} &\mathcal{E}_{\mathrm{YM}}(A) \geq |\kappa|, \\ &\mathcal{E}_{\mathrm{YM}}(A) = \pm\kappa \iff F_A^{\pm} = 0. \end{aligned}$$

A priori estimate

$F_A^+ = 0 \implies \mathcal{E}_{\mathrm{YM}}(A) = \kappa.$

This implies weak L^2_1 compactness up to gauge transformations, in the sense of Sedlacek. Elliptic regularity implies Uhlenbeck compactness.

The Seiberg-Witten equations

$$\begin{aligned} F_A^+ - \tfrac{1}{4}(\phi\otimes\phi^*)_0 &= 0, \\ \not{\partial}_A\phi &= 0. \end{aligned}$$

P is a $\mathrm{U}(1)$ bundle with connection A and associated line bundle L.

ϕ is a section of the twisted spinor bundle $\$\otimes L$.

Clifford multiplication identifies F_A^+ with a traceless endomorphism of $\$\otimes L$.

Define energy as the sum of squares:

$$\mathcal{E}_{\mathrm{SW}}(A,\phi) := \int_X |F_A^+ - \tfrac{1}{4}(\phi\otimes\phi^*)_0|^2 + \int_X |\not{\partial}_A\phi|^2 .$$

Energy identities for Seiberg-Witten

$$\mathcal{E}_{\mathrm{SW}}(A,\phi) := \int_X |F_A^+ - \tfrac{1}{4}(\phi\otimes\phi^*)_0|^2 + \int_X |\not{\partial}_A\phi|^2 .$$

The Weitzenböck formula for spinors gives

$$\int_X |\partial_A\phi|^2 = \int_X \left(|\nabla_A\phi|^2 + \tfrac{1}{4}s\,|\phi|^2 - \tfrac{1}{2}\langle\phi, F_A^+\cdot\phi\rangle\right).$$

Expanding out $\mathcal{E}_{\mathrm{SW}}$ gives

$$\begin{aligned} \mathcal{E}_{\mathrm{SW}} = &\int_X \left(|F_A^+|^2 + \tfrac{1}{16}\,|(\phi\otimes\phi^*)_0|^2 + \cancel{\tfrac{1}{2}\langle F_A^+, (\phi\otimes\phi^*)_0\rangle}\right) \\ &+ \int_X \left(|\nabla_A\phi|^2 + \tfrac{1}{4}s\,|\phi|^2 - \cancel{\tfrac{1}{2}\langle\phi, F_A^+\cdot\phi\rangle}\right) \end{aligned}$$

$$\mathcal{E}_{\mathrm{SW}} = \int_X \left(|F_A^+|^2 + |\nabla_A\phi|^2 + \tfrac{1}{32}\,|\phi|^4 + \tfrac{1}{4}s\,|\phi|^2\right)$$

Completing the square

$$\mathcal{E}_{\mathrm{SW}} = \int_X \left(|F_A^+|^2 + |\nabla_A \phi|^2 + \tfrac{1}{32}|\phi|^4 + \tfrac{1}{4}s|\phi|^2\right).$$

Problem When scalar curvature is negative, the term $\int_X \frac{1}{4}s|\phi|^2$ could be large and negative.

Solution Complete the square:

$$\begin{aligned}\mathcal{E}_{\mathrm{SW}} &= \int_X \left(|F_A^+|^2 + |\nabla_A\phi|^2 + \tfrac{1}{32}\left(|\phi|^2 + 4s\right)^2 - \tfrac{1}{2}s^2\right)\\ &= \int_X \left(\tfrac{1}{2}|F_A|^2 + |\nabla_A\phi|^2 + \tfrac{1}{32}\left(|\phi|^2+4s\right)^2\right) - \kappa - \int_X \tfrac{1}{2}s^2.\end{aligned}$$

Compactness for Seiberg-Witten

A priori estimate

$$\mathcal{E}_{\mathrm{SW}} = 0 \iff$$
$$\int_X \left(\tfrac{1}{2}|F_A|^2 + |\nabla_A\phi|^2 + \tfrac{1}{32}\left(|\phi|^2+4s\right)^2\right) = \kappa + \int_X \tfrac{1}{2}s^2.$$

The left hand side is a sum of positive terms. The quartic term is essentially $\|\phi\|_{L^4}^4$.

The right hand side depends only on the (fixed) topology of the bundle, and geometry of the manifold.

Since the gauge group is abelian, elementary Hodge theory yields genuine L_1^2 bounds on A and ϕ, and elliptic bootstrapping implies compactness.

Energy identities for Vafa-Witten

We emulate the standard approach:

$$\mathcal{E}_{\mathrm{VW}}(C,A,B) := \tfrac{1}{2}\|d_A C + d_A^* B\|^2 + \|F_A^+ - \tfrac{1}{4}[B\times B] - \tfrac{1}{2}[C,B]\|^2$$

$$\begin{aligned}&= \tfrac{1}{2}\|d_A C\|^2 + \tfrac{1}{2}\|d_A^* B\|^2 + \|F_A^+ - \tfrac{1}{4}[B\times B]\|^2 + \tfrac{1}{4}\|[C,B]\|^2\\ &\quad + \int_X \left(\langle d_A C, d_A^* B\rangle - \langle F_A^+, [C,B]\rangle\right) + \int_X \tfrac{1}{4}\langle [B\times B], [C,B]\rangle.\end{aligned}$$

The bottom line cancels since

$$\langle F_A^+, [C,B]\rangle = \langle [F_A^+, C], B\rangle = \langle d_A d_A C, B\rangle,$$

and the Jacobi identity implies

$$[[B\times B]\cdot B] = 0.$$

Simplification

Thus, assuming that the base manifold X is closed, we have the identity

$$\mathcal{E}_{\mathrm{VW}} = \tfrac{1}{2}\|d_A C\|^2 + \tfrac{1}{2}\|d_A^* B\|^2 + \|F_A^+ - \tfrac{1}{4}[B\times B]\|^2 + \tfrac{1}{4}\|[C,B]\|^2.$$

This is a different sum of squares, equivalent equations are

$$F_A^+ - \tfrac{1}{4}[B\times B] = 0, \qquad [C,B] = 0,$$
$$d_A^* B = 0, \qquad d_A C = 0.$$

These equations are linear in C. The interesting nonlinear part with B decouples. WLOG, set $C = 0$.

Analogy with Seiberg-Witten

The equations

$$F_A^+ = \tfrac{1}{4}[B \times B]$$
$$d_A^* B = 0$$

These equations say that B has a harmonic square root, if we interpret "B^2" $= [B \times B]$.

$$B = \text{“}2\sqrt{F_A^+}\text{”}$$
$$d_A^* B = 0 \quad (\Rightarrow d_A B = 0)$$

Contrast this with the Seiberg-Witten equations

$$F_A^+ - (\phi \otimes \phi^*)_0 = 0$$
$$\not{\partial}_A \phi = 0$$

The Weitzenböck formula

$$\tfrac{1}{2}\|d_A^* B\|^2 = \tfrac{1}{4}\|\nabla_A B\|^2 + \int_X \big(\tfrac{1}{2}\langle B, [F_A^+ \times B]\rangle + $$
$$+ \left(\tfrac{1}{12}s - \tfrac{1}{2}W^+\right) \cdot \langle B \otimes B\rangle \big).$$

$$\mathcal{E}_{\mathrm{VW}}(0, A, B) = \|F_A^+ - \tfrac{1}{4}[B \times B]\|^2 + \tfrac{1}{2}\|d_A^* B\|^2 .$$

Once again, the cross-term miraculously cancels:

$$\mathcal{E}_{\mathrm{VW}} = \|F_A^+\|^2 + \tfrac{1}{16}\|[B \times B]\|^2 + \tfrac{1}{4}\|\nabla_A B\|^2 +$$
$$\int_X \tfrac{1}{2}\Big(\cancel{\langle B, [F_A^+ \times B]\rangle - \langle F_A^+, [B \times B]\rangle} \Big) + \int_X \left(\tfrac{1}{12}s - \tfrac{1}{2}W^+\right) \cdot \langle B \otimes B\rangle .$$

Vanishing theorem

The Vafa-Witten equations (with $C = 0$) are equivalent to

$$0 = \tfrac{1}{2}\|F_A^+\|^2 + \tfrac{1}{4}\|\nabla_A B\|^2 + \tfrac{1}{16}\|[B \times B]\|^2 +$$
$$+ \int_X \left(\tfrac{1}{12}s - \tfrac{1}{2}W^+\right) \cdot \langle B \otimes B\rangle .$$

If furthermore the curvature part is positive semi-definite, then M must be Kähler, hyper-Kähler, or $b^+ = 0$, and the equations decouple further to

$$F_A^+ = 0 \qquad \nabla_A B = 0 \qquad [B \times B] = 0.$$

Kähler manifolds

Let (M, g, ω) be a closed Kähler manifold. The equations

$$F_A^+ - \tfrac{1}{4}[B \times B] = 0$$
$$d_A^* B = 0$$

reduce to

$$\beta \in \Omega^{2,0}(X; \mathfrak{g}_P \otimes \mathbb{C}), \qquad B = \beta - \beta^*, \qquad \bar{\partial}_A \beta = 0,$$

$$\omega \wedge iF_A + \tfrac{1}{2}[\beta \wedge \beta^*] = 0.$$

Note the extra symmetry $\beta \mapsto e^{i\theta}\beta$ for θ constant.

Semistability

We say a subbundle $E' \subset E$ is β-*invariant* if

$$\beta(E') \subset E' \otimes K.$$

Suppose E is a Hermitian vector bundle, A is a holomorphic connection on E, the traceless part of F_A is F_A^0, and $\beta \in \Omega^{2,0}(\mathrm{End}(E))$ satisfies

$$\omega \wedge iF_A^0 + \tfrac{1}{2}\left[\beta \wedge \beta^*\right] = 0,$$

then E is β-semistable.

Failure of a priori bound

Recall that our quartic term $\left|[B\,\mathbf{.}\,B]\right|^2$ is only positive *semi*-definite. If it were positive-definite, then it would dominate the curvature part, and the identity

$$0 = \tfrac{1}{2}\left\|F_A\right\|^2 + \tfrac{1}{4}\left\|\nabla_A B\right\|^2 + \tfrac{1}{16}\left\|[B \times B]\right\|^2 + \\ + \int_X \left(\tfrac{1}{12}s - \tfrac{1}{2}W^+\right) \cdot \langle B \otimes B\rangle - \kappa$$

would yield a priori bounds on $\|F_A\|$, $\|\nabla_A B\|$, and $\|B\|_{L^4}$. Instead, we get no such bounds since $\left|[B \times B]\right|^2$ could vanish while the curvature terms go to $-\infty$ unchecked.

Abelian solutions

For the abelian group $G = \mathrm{U}(1)$, the equations with $C = 0$ reduce to

$$F_A^+ = 0, \quad d_A^* B = 0.$$

On the trivial bundle, $A = 0$, $B \in \mathcal{H}^{2,+}$ harmonic is a family of unbounded solutions whenever $b^+ > 0$.

Local behavior of B

Consider hyperbolic space $\mathbb{H}^4$ of constant sectional curvature -1, with A the trivial connection, and

$$B = \frac{i\left(e^{01} + e^{23}\right)}{\cosh^4(t/2)},$$

where t is a radial coordinate, and $\{e^i\}$ is a orthonormal and conformally flat coframe. Computation shows that $d^* B = 0$, so this is a solution.

Scaling the metric

Although these equations are not conformally invariant, there is a scaling law.

For all $\eta \in \mathbb{R}$, the space of solutions is invariant under

$$(C, A, B, g) \mapsto (e^{-\eta} C, A, e^{\eta} B, e^{2\eta} g).$$

Thus rescaling the metric can be absorbed by this rescaling of B and C.

Curvature and the width of B

If we attempt to shrink our abelian example for B on $\mathbb{H}^4$, we must scale the metric appropriately, causing the scalar curvature to blow up.

Given the natural scalings

$$\text{length} \sim e^{\eta}$$
$$\text{curvature} \sim e^{-2\eta}$$

$$B = \frac{i\left(e^{01} + e^{23}\right)}{\cosh^4(e^{-\eta} t/2)},$$

but the rescaled curvature is $e^{-2\eta}$. It's reasonable to suspect that B cannot concentrate below this length scale when curvature is fixed. This is a heuristic for Uhlenbeck compactness: B can't bubble.

More Weitzenböck

For a more concrete application of the width heuristic, consider the following identity for solutions:

$$\tfrac{1}{8}\Delta |B|^2 + \tfrac{1}{4}|\nabla_A B|^2 + \tfrac{1}{8}|[B \times B]|^2 = \langle B \cdot (-\tfrac{1}{12}s + \tfrac{1}{2}W^+)\, B\rangle$$

In particular,

$$\Delta |B|^2 \le \lambda |B|^2$$

where λ depends on curvature.

With slightly more work,

$$\Delta |B| \le \lambda |B|$$

Bounding B

Thanks to a mean-value inequality due to Morrey

$$\Delta |B| \le \lambda |B| \implies \|B\|_{L^\infty} \le c\, \|B\|_{L^2}$$

Thus

$$\|F_A^+\|_{L^\infty} = \|\tfrac{1}{4}[B \times B]\|_{L^\infty} \le c' \, \|B\|_{L^2}^2 .$$

Assuming a bound on $\|B\|_{L^2}$, we get bounds on $\|F_A^+\|_{L^\infty}$ and $\|B\|_{L^\infty}$.

If $|[B \times B]|^2$ were positive-definite, such bounds would follow automatically from a priori estimates plus maximum principle.

Feehan-Leness program for $\mathrm{PU}(2)$

Only major property distinguishing $\mathrm{PU}(2)$ monopoles and Vafa-Witten equations is $|[B \times B]|^2$ being semi-definite. Their analytic framework extends to give:

- Slice theorem
- Elliptic estimates
- Removal of singularities
- Compactness (almost!)

Compactness requires bounds on $\|F_A^+\|_{L^\infty}$ and $\|B\|_{L^\infty}$.

Truncated Vafa-Witten moduli space

$$\mathcal{M}^b_{\mathrm{VW},k} := \{[0, A, B] \in \mathcal{M}_{\mathrm{VW},k} \mid \|B\|_{L^2} \le b\}, \ b \in \mathbb{R}$$

- $\mathcal{M}^b_{\mathrm{VW},k} \subset \mathcal{M}^{b'}_{\mathrm{VW},k}$ for $b \le b'$.
- $\mathcal{M}^0_{\mathrm{VW},k} = \mathcal{M}_{\mathrm{ASD},k}$.
- $\mathcal{M}^b_{\mathrm{VW},k} = \varnothing$ for $b < 0$ or $k < -cb^4$

Each $\mathcal{M}^b_{\mathrm{VW},k}$ has an Uhlenbeck compactification $\overline{\mathcal{M}}^b_{\mathrm{VW},k}$.

A partial compactification is given by $\overline{\mathcal{M}}_{\mathrm{VW},k} := \bigcup_{b \in \mathbb{R}} \overline{\mathcal{M}}^b_{\mathrm{VW},k}$.

Kuranishi complex

The Kuranishi complex for an instanton $A \in \mathcal{A}_P$ is

$$0 \to \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^+} \Omega^{2,+}(\mathfrak{g}_P) \to 0$$

with (harmonic) cohomology

$$\mathcal{H}^0_A \qquad \mathcal{H}^1_A \qquad \mathcal{H}^2_A$$

- $\mathcal{H}^0_A$ is the infinitesimal stabilizer (vanishes for irreducibles)
- $\mathcal{H}^1_A$ is the tangent space of $\mathcal{M}_{\mathrm{ASD}}$
- $\mathcal{H}^2_A$ is the obstruction to transversality

Index theorem

$$0 \to \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^+} \Omega^{2,+}(\mathfrak{g}_P) \to 0$$

$$\mathcal{H}^0_A \qquad \mathcal{H}^1_A \qquad \mathcal{H}^2_A$$

We will always assume A is irreducible, so $\mathcal{H}^0_A = 0$.

By the index theorem,

$$\dim \mathcal{H}^1_A = d + c \qquad \dim \mathcal{H}^2_A = c$$

where d is the expected dimension $8k - 3(1 - b_1 + b^+)$ for $G = \mathrm{SU}(2)$.

For generic metrics, $c = 0$, and a neighborhood of $[A] \in \mathcal{M}_{\mathrm{ASD}}$ is modeled by $\mathcal{H}^1_A$ with dimension d.

Inverse function theorem

Consider

$$\dim \mathcal{H}_A^1 = d + 1 \qquad \dim \mathcal{H}_A^2 = 1$$

The differential

$$\Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^+} \Omega^{2,+}(\mathfrak{g}_P)$$

is no longer surjective. For $a \in H_A^1$,

$$F_{A+a}^+ = F_A^+ + d_A^+ a + \tfrac{1}{2}[a \wedge a]^+ = \tfrac{1}{2}[a \wedge a]^+ .$$

Can use inverse function theorem to find $\tilde{a}$ with $F_{A+\tilde{a}}^+ = 0$ when $\frac{1}{2}[a \wedge a]^+ \perp \mathcal{H}_A^2$.

Quadratic model

$$\dim \mathcal{H}_A^1 = d + 1 \qquad \dim \mathcal{H}_A^2 = 1$$

Fix $\omega \in \mathcal{H}_A^2$, with $\omega \neq 0$. Define a quadratic form $q(a)$ on $\mathcal{H}_A^1$ by

$$q(a) := \int_X \left\langle \tfrac{1}{2}[a \wedge a]^+ \cdot \omega \right\rangle = 0.$$

If q is nondegenerate, a neighborhood of $[A] \in \mathcal{M}_{\text{ASD}}$ is modeled on $q(a) = 0$.

Vafa-Witten quadratic model

$$\int_X \langle [a \times b] \cdot \alpha_1 \rangle = 0,$$
$$\vdots$$
$$\int_X \langle [a \times b] \cdot \alpha_{d+1} \rangle = 0,$$
$$\int_X \left\langle \tfrac{1}{2}[a \wedge a]^+ - \tfrac{1}{4}[b \times b] \cdot \omega \right\rangle = 0.$$

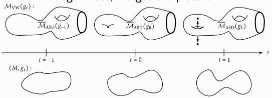
When nondegenerate, first equations say $a = 0$ or $b = 0$.

Perturbing the metric

Consider a perturbation of conformal structure $m \in \Omega^0(X; \text{Hom}(\Lambda^{2,-}V^*, \Lambda^{2,+}V^*))$, and the parameterized family tm, $t \in \mathbb{R}$. We get an extra term

$$\int_X \left\langle \left(\tfrac{1}{2}[a \wedge a]^+ - \tfrac{1}{4}[b \times b] - tmF_A^- \right) \cdot \omega \right\rangle = 0.$$

When nondegenerate, we get the picture



Gluing theory and future work

Following the work of Donaldson and Taubes, we can construct approximate solutions by grafting concentrated instantons.

The obstruction to repairing the graft to obtain a genuine solution can be approximated by a quadratic map.

Assuming nondegeneracy, we obtain quadratic models of the Uhlenbeck ends of the moduli space.

We hope to gain insight into the lingering compactness issues by studying these models.

New nonlinear estimates will hopefully lead to better a priori bounds.

IT'S OVER!

- Thanks for listening!