Some Analytic Aspects of Vafa-Witten Twisted $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

(thesis work under Tom Mrowka)

Ben Mares

September 8, 2012

Overview of 4D differential topology

- Motivating problem: Understand classification of smooth structures on (oriented) four-manifolds.
- Primary tool: Four-manifold invariants. (If invariants disagree, then smooth structures are distinct.)
- Invariants arise from studying PDEs, depending on extra structure (principal bundle, Riemannian metric, etc.)

Examples

- deRham theory $d\omega = 0$.
  Quotient by exact forms
  Invariant: Betti numbers. Only sees homotopy information.
- Gauge theory. (Various nonlinear PDEs)
  Quotient by gauge group
  Invariants: Donaldson invariants, Seiberg-Witten invariants.
  More on this later.

Examples

- deRham theory $d\omega = 0$.
  Quotient by exact forms
  Invariant: Betti numbers. Only sees homotopy information.

- Gauge theory. (Various nonlinear PDEs)
  Quotient by gauge group
  Invariants: Donaldson invariants, Seiberg-Witten invariants.
  More on this later.

Goal

Establish new invariants using Vafa-Witten equations.

- Currently only conjecturally defined.
- For many Kähler manifolds they have been "computed" by algebraic methods, and the answers satisfy the conjectured properties.
- We want to prove that these invariants exist for any oriented Riemannian manifold.
- We construct a partial Uhlenbeck compactification.
- Many issues remain unresolved before invariants become rigorous.

Goal

Establish new invariants using Vafa-Witten equations.

- Currently only conjecturally defined.
- For many Kähler manifolds they have been "computed" by algebraic methods, and the answers satisfy the conjectured properties.
- We want to prove that these invariants exist for any oriented Riemannian manifold.
- We construct a partial Uhlenbeck compactification.
- Many issues remain unresolved before invariants become rigorous.

Goal

Establish new invariants using Vafa-Witten equations.

- Currently only conjecturally defined.
- For many Kähler manifolds they have been "computed" by algebraic methods, and the answers satisfy the conjectured properties.
- We want to prove that these invariants exist for any oriented Riemannian manifold.
- We construct a partial Uhlenbeck compactification.
- Many issues remain unresolved before invariants become rigorous.

Goal

Establish new invariants using Vafa-Witten equations.

- Currently only conjecturally defined.
- For many Kähler manifolds they have been "computed" by algebraic methods, and the answers satisfy the conjectured properties.
- We want to prove that these invariants exist for any oriented Riemannian manifold.
- We construct a partial Uhlenbeck compactification.
- Many issues remain unresolved before invariants become rigorous.

Goal

Establish new invariants using Vafa-Witten equations.

- Currently only conjecturally defined.
- For many Kähler manifolds they have been "computed" by algebraic methods, and the answers satisfy the conjectured properties.
- We want to prove that these invariants exist for any oriented Riemannian manifold.
- We construct a partial Uhlenbeck compactification.
- Many issues remain unresolved before invariants become rigorous.

Why care?

- Understanding smooth structures
- Relates to the Euler characteristics of moduli spaces
- Connections with mathematical physics
- Number theory: Are the generating functions modular forms?

Review of geometry in four dimensions

Possibly the most prominent feature of four-dimensional geometry:

$$1 \longrightarrow \{\pm I\} \longrightarrow \mathrm{SO}(4) \longrightarrow \mathrm{SO}(3) \times \mathrm{SO}(3) \longrightarrow 1$$

If $\dim V = 4$, then Hodge star $\star : \Lambda^2 V \to \Lambda^2 V$ is self-adjoint, with $\star^2 = 1$, so eigenvalues of $\star$ must be ± 1.

$$\Lambda^+ V = \mathrm{span}\{e^0 \wedge e^1 + e^2 \wedge e^3, e^0 \wedge e^2 + e^3 \wedge e^1, e^0 \wedge e^3 + e^1 \wedge e^2\},$$

$$\Lambda^- V = \mathrm{span}\{e^0 \wedge e^1 - e^2 \wedge e^3, e^0 \wedge e^2 - e^3 \wedge e^1, e^0 \wedge e^3 - e^1 \wedge e^2\}.$$

Review of geometry in four dimensions

Possibly the most prominent feature of four-dimensional geometry:

$$1 \longrightarrow \{\pm I\} \longrightarrow \mathrm{SO}(4) \longrightarrow \mathrm{SO}(3) \times \mathrm{SO}(3) \longrightarrow 1$$

If $\dim V = 4$, then Hodge star $\star : \Lambda^2 V \to \Lambda^2 V$ is self-adjoint, with $\star^2 = 1$, so eigenvalues of $\star$ must be ± 1.

$$\Lambda^+ V = \mathrm{span}\Big\{e^0 \wedge e^1 + e^2 \wedge e^3, e^0 \wedge e^2 + e^3 \wedge e^1, e^0 \wedge e^3 + e^1 \wedge e^2\Big\},$$

$$\Lambda^- V = \mathrm{span}\Big\{e^0 \wedge e^1 - e^2 \wedge e^3, e^0 \wedge e^2 - e^3 \wedge e^1, e^0 \wedge e^3 - e^1 \wedge e^2\Big\}.$$

Review of geometry in four dimensions

$$\Lambda^2 V = \Lambda^+ V \oplus \Lambda^- V$$
$$6 = 3 + 3$$

$$1 \longrightarrow \{\pm I\} \longrightarrow \mathrm{SO}(V) \longrightarrow \mathrm{SO}(\Lambda^+ V) \times \mathrm{SO}(\Lambda^- V) \longrightarrow 1$$

$\Lambda^+ V$ is an oriented three-dimensional vector space associated to any 4-D oriented Euclidean V.

Review of geometry in four dimensions

$$\begin{aligned} \Lambda^2 V &= \Lambda^+ V \oplus \Lambda^- V \\ 6 &= \ 3 \ + \ 3 \end{aligned}$$

$$1 \longrightarrow \{\pm I\} \longrightarrow \mathrm{SO}(V) \longrightarrow \mathrm{SO}(\Lambda^+ V) \times \mathrm{SO}(\Lambda^- V) \longrightarrow 1$$

$\Lambda^+ V$ is an oriented three-dimensional vector space associated to any 4-D oriented Euclidean V.

Review of geometry in four dimensions

$$\begin{aligned} \Lambda^2 V &= \Lambda^+ V \oplus \Lambda^- V \\ 6 &= \quad 3 \quad + \quad 3 \end{aligned}$$

$$1 \longrightarrow \{\pm I\} \longrightarrow \mathrm{SO}(V) \longrightarrow \mathrm{SO}(\Lambda^+ V) \times \mathrm{SO}(\Lambda^- V) \longrightarrow 1$$

$\Lambda^+ V$ is an oriented three-dimensional vector space associated to any 4-D oriented Euclidean V.

The cross product

Let X be an oriented Riemannian four-manifold. For any $x \in X$, take $V = T_x^* X$.

- Choose an oriented orthonormal basis $\{e^0, e^1, e^2, e^3\}$.
- An oriented orthonormal basis for $\Lambda^+ T_x^* X$ is

$$\sigma^1 = e^0 \wedge e^1 + e^2 \wedge e^3,$$
$$\sigma^2 = e^0 \wedge e^2 + e^3 \wedge e^1,$$
$$\sigma^3 = e^0 \wedge e^3 + e^1 \wedge e^2.$$

- Define the cross product on $\Lambda^+ T_x^* X$ via $\{\sigma^i\}$ components, so $\sigma^1 \times \sigma^2 = \sigma^3$.

The cross product

Let X be an oriented Riemannian four-manifold. For any $x \in X$, take $V = T_x^* X$.

- Choose an oriented orthonormal basis $\{e^0, e^1, e^2, e^3\}$.
- An oriented orthonormal basis for $\Lambda^+ T_x^* X$ is

$$\sigma^1 = e^0 \wedge e^1 + e^2 \wedge e^3,$$
$$\sigma^2 = e^0 \wedge e^2 + e^3 \wedge e^1,$$
$$\sigma^3 = e^0 \wedge e^3 + e^1 \wedge e^2.$$

- Define the cross product on $\Lambda^+ T_x^* X$ via $\{\sigma^i\}$ components, so $\sigma^1 \times \sigma^2 = \sigma^3$.

The cross product

Let X be an oriented Riemannian four-manifold. For any $x \in X$, take $V = T_x^* X$.

- Choose an oriented orthonormal basis $\{e^0, e^1, e^2, e^3\}$.
- An oriented orthonormal basis for $\Lambda^+ T_x^* X$ is

$$\sigma^1 = e^0 \wedge e^1 + e^2 \wedge e^3,$$
$$\sigma^2 = e^0 \wedge e^2 + e^3 \wedge e^1,$$
$$\sigma^3 = e^0 \wedge e^3 + e^1 \wedge e^2.$$

- Define the cross product on $\Lambda^+ T_x^* X$ via $\{\sigma^i\}$ components, so $\sigma^1 \times \sigma^2 = \sigma^3$.

The de Rham complex

Inside the de Rham complex

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \xrightarrow{d} \Omega^4 \to 0$$

is the subcomplex

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^{2,+} \to 0.$$

$$b^0 \qquad b^1 \qquad b^+$$

Given a principal bundle P with connection A,

$$0 \to \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^+} \Omega^{2,+}(\mathfrak{g}_P) \to 0.$$

The double-composition is

$$d_A^+ \circ d_A = \left[F_A^+, \bullet\right].$$

This defines a complex when $F_A^+ = 0$.

The de Rham complex

Inside the de Rham complex

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \xrightarrow{d} \Omega^4 \to 0$$

is the subcomplex

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^{2,+} \to 0.$$

$$b^0 \qquad b^1 \qquad b^+$$

Given a principal bundle P with connection A,

$$0 \to \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^+} \Omega^{2,+}(\mathfrak{g}_P) \to 0.$$

The double-composition is

$$d_A^+ \circ d_A = \left[F_A^+, \bullet\right].$$

This defines a complex when $F_A^+ = 0$.

The de Rham complex

Inside the de Rham complex

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \xrightarrow{d} \Omega^4 \to 0$$

is the subcomplex

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^{2,+} \to 0.$$

$$b^0 \qquad b^1 \qquad b^+$$

Given a principal bundle P with connection A,

$$0 \to \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^+} \Omega^{2,+}(\mathfrak{g}_P) \to 0.$$

The double-composition is

$$d_A^+ \circ d_A = \left[F_A^+, \bullet\right].$$

This defines a complex when $F_A^+ = 0$.

Anti-self-dual equation

People often study the equation $F_A^+ = 0$.

It is called the anti-self-dual equation since $F_A^+ = 0 \iff F_A = F_A^-$.

Solutions arise as absolute minimizers of $\|F_A\|_{L^2}$.

If $g \in \mathcal{G}_P$ is a gauge transformation, then

$$F_{g(A)}^+ = g \cdot F_A^+ \cdot g^{-1},$$

so $\mathcal{G}_P$ preserves solutions to $F_A^+ = 0$. The moduli space

$$\mathcal{M}_{ASD} = \{[A] \in \mathcal{A}_P/\mathcal{G}_P | F_A^+ = 0\}$$

is finite-dimensional. Roughly speaking, it defines a homology class in $\mathcal{A}_P/\mathcal{G}_P$. This leads to Donaldson invariants.

Anti-self-dual equation

People often study the equation $F_A^+ = 0$.

It is called the anti-self-dual equation since $F_A^+ = 0 \iff F_A = F_A^-$.

Solutions arise as absolute minimizers of $\|F_A\|_{L^2}$.

If $g \in \mathcal{G}_P$ is a gauge transformation, then

$$F_{g(A)}^+ = g \cdot F_A^+ \cdot g^{-1},$$

so $\mathcal{G}_P$ preserves solutions to $F_A^+ = 0$. The moduli space

$$\mathcal{M}_{\text{ASD}} = \{[A] \in \mathcal{A}_P/\mathcal{G}_P | F_A^+ = 0\}$$

is finite-dimensional. Roughly speaking, it defines a homology class in $\mathcal{A}_P/\mathcal{G}_P$. This leads to Donaldson invariants.

Anti-self-dual equation

People often study the equation $F_A^+ = 0$.

It is called the anti-self-dual equation since $F_A^+ = 0 \iff F_A = F_A^-$.

Solutions arise as absolute minimizers of $\|F_A\|_{L^2}$.

If $g \in \mathcal{G}_P$ is a gauge transformation, then

$$F_{g(A)}^+ = g \cdot F_A^+ \cdot g^{-1},$$

so $\mathcal{G}_P$ preserves solutions to $F_A^+ = 0$. The moduli space

$$\mathcal{M}_{\text{ASD}} = \{[A] \in \mathcal{A}_P/\mathcal{G}_P | F_A^+ = 0\}$$

is finite-dimensional. Roughly speaking, it defines a homology class in $\mathcal{A}_P/\mathcal{G}_P$. This leads to Donaldson invariants.

Motivating question for studying Vafa-Witten

- What is the Euler characteristic of the ASD moduli space $\mathcal{M}_{\text{ASD}}$?
  - Is this question meaningful?
    - Singularities
    - Metric dependence
    - Choice of compactification (if any)

Motivating question for studying Vafa-Witten

- What is the Euler characteristic of the ASD moduli space $\mathcal{M}_{\mathrm{ASD}}$?
  - Is this question meaningful?
    - Singularities
    - Metric dependence
    - Choice of compactification (if any)

Comparison with Donaldson invariants

Write $\mathcal{M}_{\mathrm{ASD}}$ as a zero set:

$$\mathcal{M}_{\mathrm{ASD}} = \{[A] \in \mathcal{A}/\mathcal{G} \mid F_A^+ = 0\}.$$

If $\dim(\mathcal{M}_{\mathrm{ASD}}) = 0$, then the Donaldson invariant is a signed count $\#\mathcal{M}_{\mathrm{ASD}}$.

Analogy with polynomials:

- Let $M = \{x | x^2 - c = 0\}$. How many points are in m?
  - Signed count $\#M$ gives $+1 - 1 = 0$.
    - Generically well-defined on $\mathbb{R}$.
  - Unsigned count $\chi(M)$ gives $1 + 1 = 2$.
    - Generically well-defined on $\mathbb{C}$.

Comparison with Donaldson invariants

Write $\mathcal{M}_{\text{ASD}}$ as a zero set:

$$\mathcal{M}_{\text{ASD}} = \{[A] \in \mathcal{A}/\mathcal{G} \,|\, F_A^+ = 0\}.$$

If $\dim(\mathcal{M}_{\text{ASD}}) = 0$, then the Donaldson invariant is a signed count $\#\mathcal{M}_{\text{ASD}}$.

Analogy with polynomials:

- Let $M = \{x|x^2 - c = 0\}$. How many points are in m?
  - Signed count $\#M$ gives $+1 - 1 = 0$.
    - Generically well-defined on $\mathbb{R}$.
  - Unsigned count $\chi(M)$ gives $1 + 1 = 2$.
    - Generically well-defined on $\mathbb{C}$.

Comparison with Donaldson invariants

Write $\mathcal{M}_{\text{ASD}}$ as a zero set:

$$\mathcal{M}_{\text{ASD}} = \left\{ [A] \in \mathcal{A}/\mathcal{G} \,\middle|\, F_A^+ = 0 \right\}.$$

If $\dim(\mathcal{M}_{\text{ASD}}) = 0$, then the Donaldson invariant is a signed count $\#\mathcal{M}_{\text{ASD}}$.

Analogy with polynomials:

- Let $M = \{x | x^2 - c = 0\}$. How many points are in m?
  - Signed count $\#M$ gives $+1 - 1 = 0$.
    - Generically well-defined on $\mathbb{R}$.
  - Unsigned count $\chi(M)$ gives $1 + 1 = 2$.
    - Generically well-defined on $\mathbb{C}$.

Comparison with Donaldson invariants

Write $\mathcal{M}_{\text{ASD}}$ as a zero set:

$$\mathcal{M}_{\text{ASD}} = \left\{[A] \in \mathcal{A}/\mathcal{G} \,\middle|\, F_A^+ = 0\right\}.$$

If $\dim(\mathcal{M}_{\text{ASD}}) = 0$, then the Donaldson invariant is a signed count $\#\mathcal{M}_{\text{ASD}}$.

Analogy with polynomials:

- Let $M = \{x | x^2 - c = 0\}$. How many points are in m?
  - Signed count $\#M$ gives $+1 - 1 = 0$.
    - Generically well-defined on $\mathbb{R}$.
  - Unsigned count $\chi(M)$ gives $1 + 1 = 2$.
    - Generically well-defined on $\mathbb{C}$.

“Complexification” of configuration space

When $\dim(\mathcal{M}_{\text{ASD}}) = 0$, we expect the signed count of the Donaldson invariant $\#\mathcal{M}_{\text{ASD}}$ to typically differ from the unsigned count “$\chi(\mathcal{M}_{\text{ASD}})$”.

In analogy with complexification, we will “double” the degrees of freedom in our configuration space by adding extra fields. This leads to an augmented moduli space $\mathcal{M}_{\text{VW}}$ with

$$\mathcal{M}_{\text{ASD}} \subset \mathcal{M}_{\text{VW}}.$$

“Complexification” of configuration space

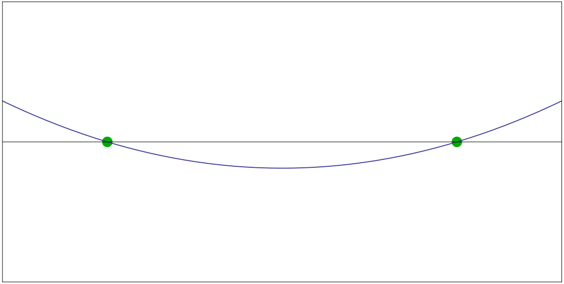
When $\dim(\mathcal{M}_{\text{ASD}}) = 0$, we expect the signed count of the Donaldson invariant $\#\mathcal{M}_{\text{ASD}}$ to typically differ from the unsigned count “$\chi(\mathcal{M}_{\text{ASD}})$”.

In analogy with complexification, we will “double” the degrees of freedom in our configuration space by adding extra fields. This leads to an augmented moduli space $\mathcal{M}_{\text{VW}}$ with

$$\mathcal{M}_{\text{ASD}} \subset \mathcal{M}_{\text{VW}}.$$

Counting zeroes of a section

Consider a vector bundle $V \to X$ with a section $s : X \to V$.

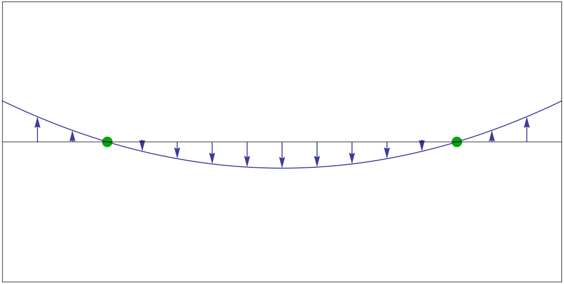


Consider s as a vector field over the zero section in the total space.

Somehow extend the vector field to the total space...

Counting zeroes of a section

Consider a vector bundle $V \to X$ with a section $s : X \to V$.

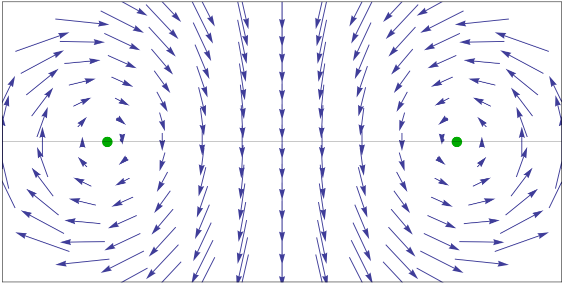


Consider s as a vector field over the zero section in the total space.

Somehow extend the vector field to the total space...

Counting zeroes of a section

Consider a vector bundle $V \to X$ with a section $s : X \to V$.

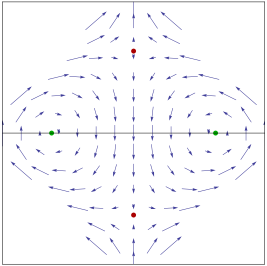


Consider s as a vector field over the zero section in the total space.

Somehow extend the vector field to the total space...

Vanishing theorem

Ideally, our extended vector field will have no additional zeroes. This is the content of a "vanishing theorem."



$\ddot{\frown}$

If a vanishing theorem holds, the zeroes of our vector field agree with the zeroes of our section, and the signed zero count of our vector field equals the unsigned zero count of our section.

Euler characteristic of $\mathcal{M}_{\text{ASD}}$

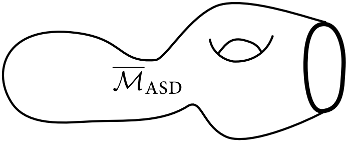
In this finite-dimensional analogy, $\mathcal{M}_{\text{ASD}}$ is the zero-set of the section, and $\mathcal{M}_{\text{VW}}$ is the zero-set of the vector field.

When a vanishing theorem holds, $\mathcal{M}_{\text{ASD}} = \mathcal{M}_{\text{VW}}$. In this case, we expect

$$\text{“}\#\mathcal{M}_{\text{VW}}\text{”} = \text{“}\chi(\mathcal{M}_{\text{ASD}})\text{.”}$$

What's wrong with $\chi(\mathcal{M}_{\text{ASD}})$?

The Poincaré-Hopf index theorem only computes the Euler characteristic of a compact manifold. Since $\mathcal{M}_{\text{ASD}}$ is non-compact, we need a compactification $\overline{\mathcal{M}}_{\text{ASD}}$:

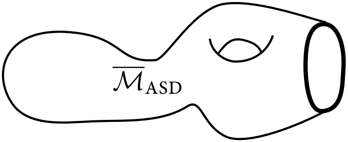


The invariant should be independent of the metric g, but different choices of g typically lead to *cobordant* compactified ASD moduli spaces $\overline{\mathcal{M}}_{\text{ASD}}(g)$.

Euler characteristic is *not* invariant under cobordism! $(\mathbb{S}^2)$

What's wrong with $\chi(\mathcal{M}_{\text{ASD}})$?

The Poincaré-Hopf index theorem only computes the Euler characteristic of a compact manifold. Since $\mathcal{M}_{\text{ASD}}$ is non-compact, we need a compactification $\overline{\mathcal{M}}_{\text{ASD}}$:

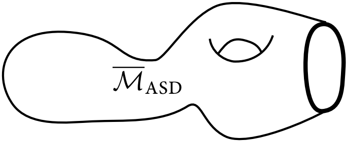


The invariant should be independent of the metric g, but different choices of g typically lead to *cobordant* compactified ASD moduli spaces $\overline{\mathcal{M}}_{\text{ASD}}(g)$.

Euler characteristic is *not* invariant under cobordism! ($\mathbb{S}^2$)

What's wrong with $\chi(\mathcal{M}_{\mathrm{ASD}})$?

The Poincaré-Hopf index theorem only computes the Euler characteristic of a compact manifold. Since $\mathcal{M}_{\mathrm{ASD}}$ is non-compact, we need a compactification $\overline{\mathcal{M}}_{\mathrm{ASD}}$:

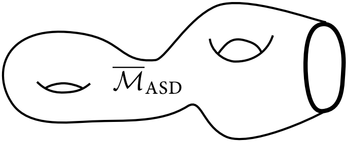


The invariant should be independent of the metric g, but different choices of g typically lead to *cobordant* compactified ASD moduli spaces $\overline{\mathcal{M}}_{\mathrm{ASD}}(g)$.

Euler characteristic is *not* invariant under cobordism! ($\mathbb{S}^2$)

What's wrong with $\chi(\mathcal{M}_{\text{ASD}})$?

The Poincaré-Hopf index theorem only computes the Euler characteristic of a compact manifold. Since $\mathcal{M}_{\text{ASD}}$ is non-compact, we need a compactification $\overline{\mathcal{M}}_{\text{ASD}}$:

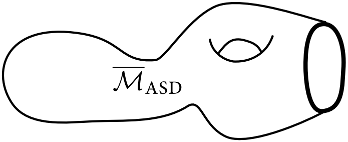


The invariant should be independent of the metric g, but different choices of g typically lead to *cobordant* compactified ASD moduli spaces $\overline{\mathcal{M}}_{\text{ASD}}(g)$.

Euler characteristic is *not* invariant under cobordism! $(\mathbb{S}^2)$

What's wrong with $\chi(\mathcal{M}_{\text{ASD}})$?

The Poincaré-Hopf index theorem only computes the Euler characteristic of a compact manifold. Since $\mathcal{M}_{\text{ASD}}$ is non-compact, we need a compactification $\overline{\mathcal{M}}_{\text{ASD}}$:

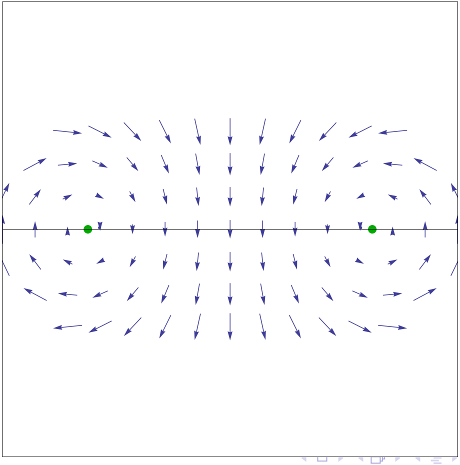


The invariant should be independent of the metric g, but different choices of g typically lead to *cobordant* compactified ASD moduli spaces $\overline{\mathcal{M}}_{\text{ASD}}(g)$.

Euler characteristic is *not* invariant under cobordism! ($\mathbb{S}^2$)

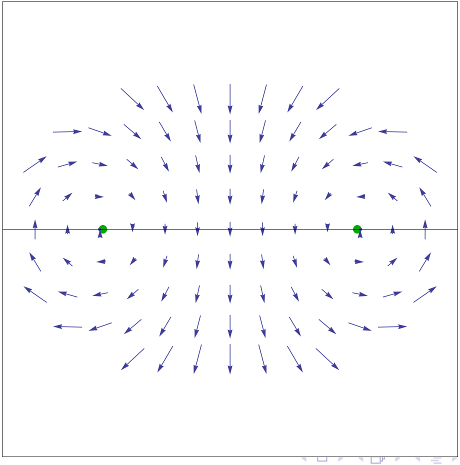
Behavior of families

Consider $0 = x^2 - c$ as c goes from positive to negative:



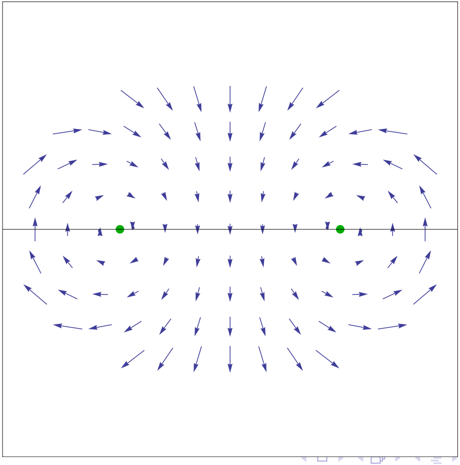
Behavior of families

Consider $0 = x^2 - c$ as c goes from positive to negative:



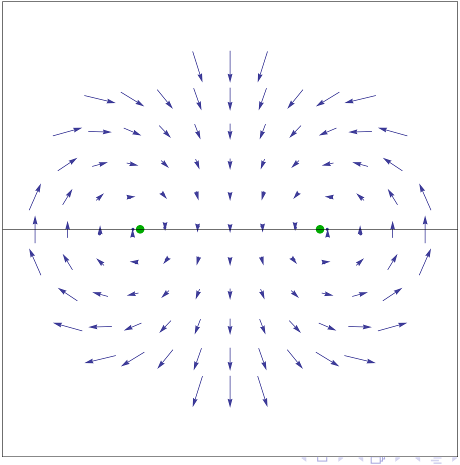
Behavior of families

Consider $0 = x^2 - c$ as c goes from positive to negative:



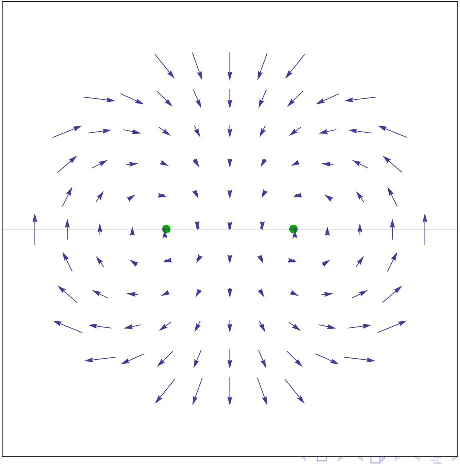
Behavior of families

Consider $0 = x^2 - c$ as c goes from positive to negative:



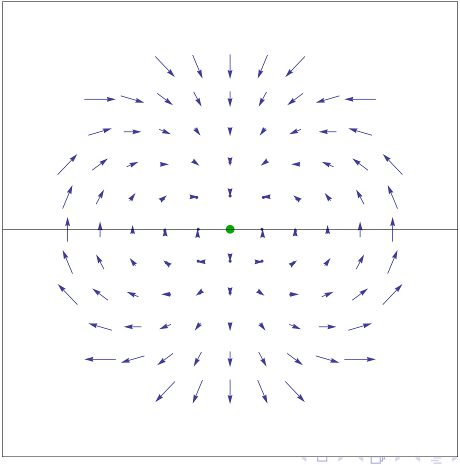
Behavior of families

Consider $0 = x^2 - c$ as c goes from positive to negative:



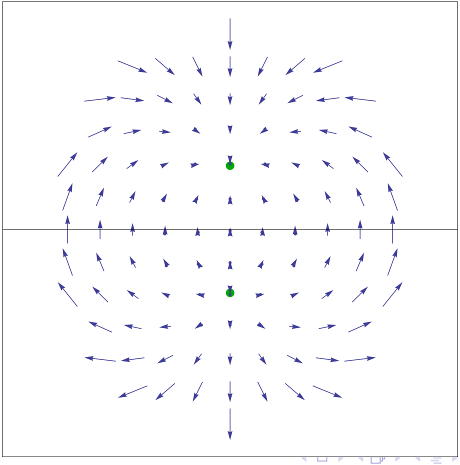
Behavior of families

Consider $0 = x^2 - c$ as c goes from positive to negative:



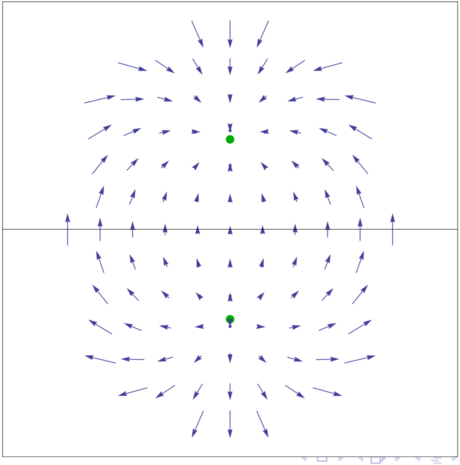
Behavior of families

Consider $0 = x^2 - c$ as c goes from positive to negative:



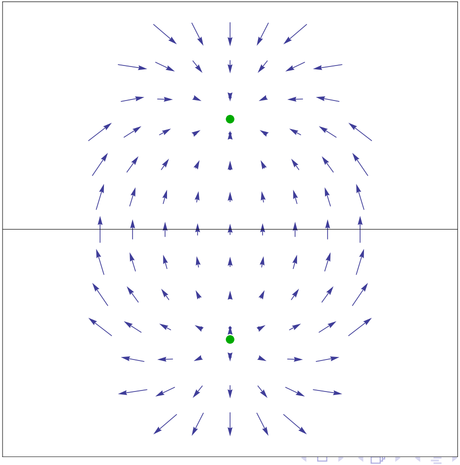
Behavior of families

Consider $0 = x^2 - c$ as c goes from positive to negative:



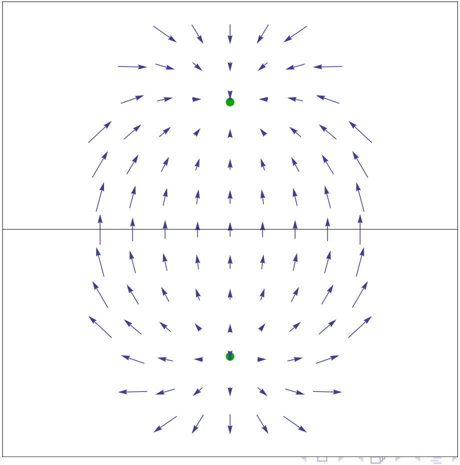
Behavior of families

Consider $0 = x^2 - c$ as c goes from positive to negative:



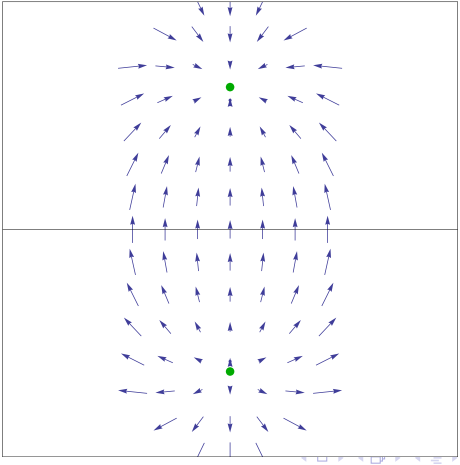
Behavior of families

Consider $0 = x^2 - c$ as c goes from positive to negative:



Behavior of families

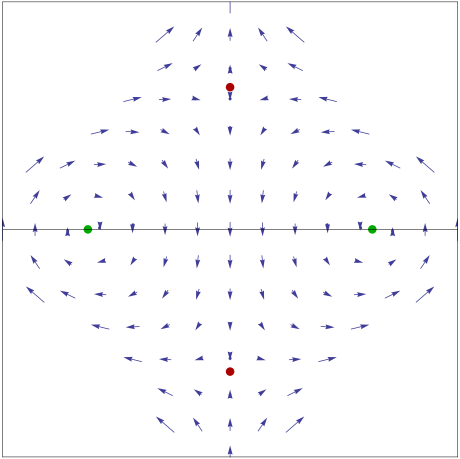
Consider $0 = x^2 - c$ as c goes from positive to negative:



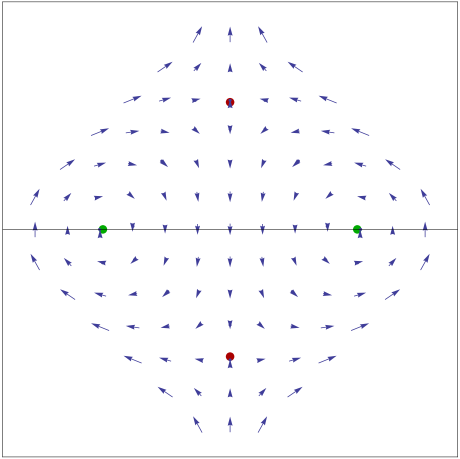
Alternative behavior

We could have extended the vector field differently.

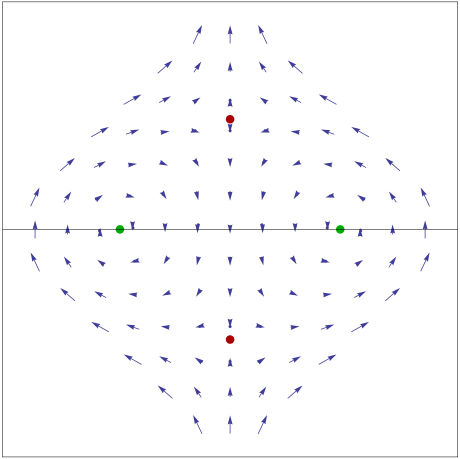
Alternative behavior



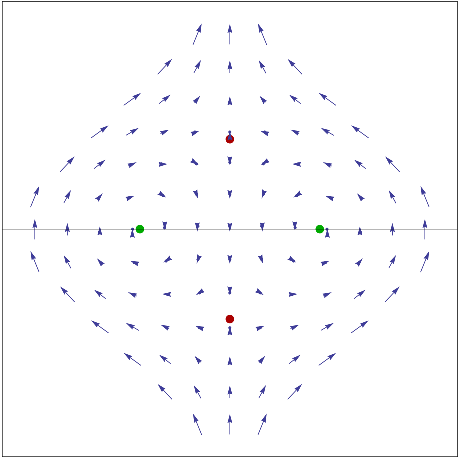
Alternative behavior



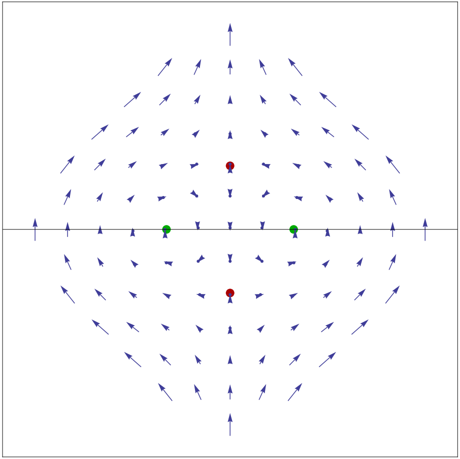
Alternative behavior



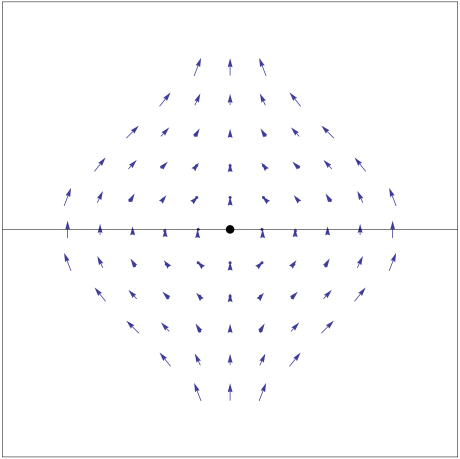
Alternative behavior



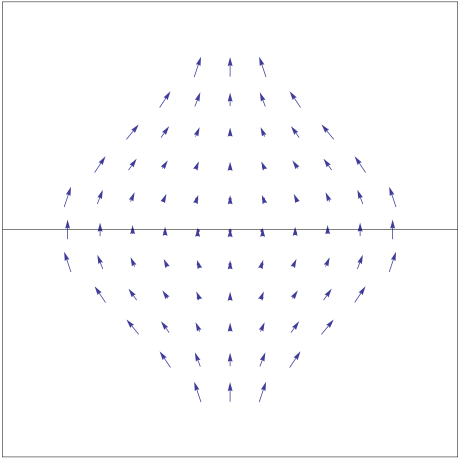
Alternative behavior



Alternative behavior

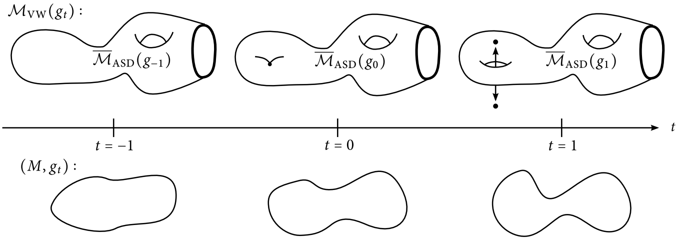


Alternative behavior



What is the purpose of extra fields?

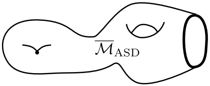
Consider a one-parameter family of metrics $\{g_t\}$ for $t \in \mathbb{R}$.



As the Euler characteristic of $\overline{\mathcal{M}}_{\mathrm{ASD}}$ changes, points in $\mathcal{M}_{\mathrm{VW}}$ should be created or destroyed to compensate.

Additional pathologies

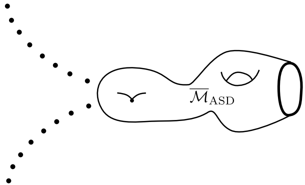
The previous picture is a fantasy. The crux of this program is to understand how to deal with pathologies.



- Sequences of solutions in $\overline{\mathcal{M}}_{\text{VW}}$ could have unbounded L^2 norms.
- Rays appear in $\overline{\mathcal{M}}_{\text{VW}}$ at reducible points of $\overline{\mathcal{M}}_{\text{ASD}}$.
- Despite having expected dimension zero, there are often manifolds of non-ASD solutions.

Additional pathologies

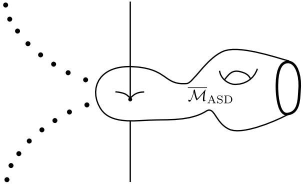
The previous picture is a fantasy. The crux of this program is to understand how to deal with pathologies.



- Sequences of solutions in $\overline{\mathcal{M}}_{\text{VW}}$ could have unbounded L^2 norms.
- Rays appear in $\overline{\mathcal{M}}_{\text{VW}}$ at reducible points of $\overline{\mathcal{M}}_{\text{ASD}}$.
- Despite having expected dimension zero, there are often manifolds of non-ASD solutions.

Additional pathologies

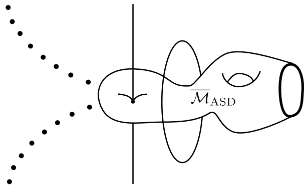
The previous picture is a fantasy. The crux of this program is to understand how to deal with pathologies.



- Sequences of solutions in $\overline{\mathcal{M}}_{\mathrm{VW}}$ could have unbounded L^2 norms.
- Rays appear in $\overline{\mathcal{M}}_{\mathrm{VW}}$ at reducible points of $\overline{\mathcal{M}}_{\mathrm{ASD}}$.
- Despite having expected dimension zero, there are often manifolds of non-ASD solutions.

Additional pathologies

The previous picture is a fantasy. The crux of this program is to understand how to deal with pathologies.



- Sequences of solutions in $\overline{\mathcal{M}}_{\text{VW}}$ could have unbounded L^2 norms.
- Rays appear in $\overline{\mathcal{M}}_{\text{VW}}$ at reducible points of $\overline{\mathcal{M}}_{\text{ASD}}$.
- Despite having expected dimension zero, there are often manifolds of non-ASD solutions

Atiyah-Jeffrey supersymmetry

There is an Atiyah-Jeffrey style supersymmetric path integral expression for the Euler characteristic of $\mathcal{M}_{\mathrm{ASD}}$.

$$“\chi(\mathcal{M}_{\mathrm{ASD}})” = “\int e^{-L}.”$$

Vafa and Witten recognized Yamron's twist of $\mathcal{N} = 4$ supersymmetric Yang-Mills as such.

They were studying $\mathcal{N} = 4$ supersymmetry in the context of S-duality.

S-duality and geometric Langlands

In this context, S-duality roughly means that the generating function

$$\sum_k \text{“}\chi(\mathcal{M}_{\text{ASD}}(k))\text{”} q^k$$

should be a modular form.

In several specific examples, they “computed” these generating functions and verified their modularity.

This Vafa-Witten theory is one of three twists of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. In 2006, Kapustin and Witten explored the relation of another such twist is to geometric Langlands. More recently, the Vafa-Witten twist has appeared in the work of Haydys and Witten on five-dimensional gauge theory.

S-duality and geometric Langlands

In this context, S-duality roughly means that the generating function

$$\sum_k \text{“}\chi(\mathcal{M}_{\mathrm{ASD}}(k))\text{”} q^k$$

should be a modular form.

In several specific examples, they “computed” these generating functions and verified their modularity.

This Vafa-Witten theory is one of three twists of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. In 2006, Kapustin and Witten explored the relation of another such twist is to geometric Langlands. More recently, the Vafa-Witten twist has appeared in the work of Haydys and Witten on five-dimensional gauge theory.

S-duality and geometric Langlands

In this context, S-duality roughly means that the generating function

$$\sum_k \text{“}\chi(\mathcal{M}_{\mathrm{ASD}}(k))\text{”} q^k$$

should be a modular form.

In several specific examples, they “computed” these generating functions and verified their modularity.

This Vafa-Witten theory is one of three twists of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. In 2006, Kapustin and Witten explored the relation of another such twist is to geometric Langlands. More recently, the Vafa-Witten twist has appeared in the work of Haydys and Witten on five-dimensional gauge theory.

Explicit example of S-duality

Consider the four-manifold $X = \mathrm{K3}$. The generating functions for $G = SU(2)$ and $\hat{G} = SO(3)$ are

$$Z_{SU(2)}(q) = \tfrac{1}{2}q^{-2}\left(\tfrac{1}{4} + 0q + 30q^2 + 3200q^3 + \cdots \right.$$
$$\left. \cdots + \tfrac{101897907561785049 75}{4}q^{16} + \cdots\right)$$

$$Z_{SO(3)}(q) = q^{-2}\left(\tfrac{1}{4} + 0q^{1/2} + 0q + 2096128q^{3/2} + \right.$$
$$+ 50356230q^2 + 679145472q^{5/2} +$$
$$\left. \cdots + \tfrac{2137997440957227092282 4975}{4}q^{16} + \cdots\right)$$

Define $q^{1/2} = e^{i\pi\tau}$. In this case, S-duality is the "modular relation"

$$Z_{SU(2)}(-1/\tau) = (2\tau)^{-12} Z_{SO(3)}(\tau).$$

Explicit example of S-duality

Consider the four-manifold $X = \mathrm{K3}$. The generating functions for $G = \mathrm{SU}(2)$ and $\hat{G} = \mathrm{SO}(3)$ are

$$Z_{\mathrm{SU}(2)}(q) = \tfrac{1}{2}q^{-2}\left(\tfrac{1}{4} + 0q + 30q^2 + 3200q^3 + \cdots \right.$$
$$\left. \cdots + \tfrac{1018979075617850497 5}{4}q^{16} + \cdots \right)$$

$$Z_{\mathrm{SO}(3)}(q) = q^{-2}\left(\tfrac{1}{4} + 0q^{1/2} + 0q + 2096128q^{3/2} + \right.$$
$$+ 50356230q^2 + 679145472q^{5/2} +$$
$$\left. \cdots + \tfrac{21379974409572270922824975}{4}q^{16} + \cdots \right)$$

Define $q^{1/2} = e^{i\pi\tau}$. In this case, S-duality is the "modular relation"

$$Z_{\mathrm{SU}(2)}(-1/\tau) = (2\tau)^{-12} Z_{\mathrm{SO}(3)}(\tau).$$

The equations

The equations

$$F_A^+ - \tfrac{1}{4}\left[B \times B\right] - \tfrac{1}{2}\left[C, B\right] = 0$$
$$d_A C + d_A^* B = 0$$

Let $P \to X^4$ be a principal bundle over an oriented Riemannian four-manifold. A *configuration* (C, A, B) consists of

- A section of the adjoint bundle $C \in \Omega^0(M; \mathfrak{g}_P)$
- A connection $A \in \mathcal{A}_P$
- An adjoint-valued self-dual two-form $B \in \Omega^{2,+}(M; \mathfrak{g}_P)$

The quadratic term

The equations

$$F_A^+ - \tfrac{1}{4}[B \times B] - \tfrac{1}{2}[C, B] = 0$$
$$d_A C + d_A^* B = 0$$

This quadratic term on $\mathfrak{g} \otimes \Lambda^{2,+}$ is the tensor product of the Lie bracket and the cross product.

Since $[,]$ is antisymmetric on $\mathfrak{g}$ and $\times$ is antisymmetric on $\Lambda^{2,+}$, their product $[B_1 \times B_2]$ is *symmetric*.

Only semi-definite

Note that $[B \times B]$ has a nontrivial kernel. Later we will see that this has dire consequences.

For example, if B has rank one

$$B = \chi \otimes \sigma^1,$$

then

$$[B \times B] = [\chi, \chi] \otimes (\sigma^1 \times \sigma^1) = 0 \otimes 0.$$

The quartic form $|[B \times B]|^2$ is only semi-definite.

Only semi-definite

Note that $[B \times B]$ has a nontrivial kernel. Later we will see that this has dire consequences.

For example, if B has rank one

$$B = \chi \otimes \sigma^1,$$

then

$$[B \times B] = [\chi, \chi] \otimes (\sigma^1 \times \sigma^1) = 0 \otimes 0.$$

The quartic form $|[B \times B]|^2$ is only semi-definite.

The standard compactness strategy

- Use energy identities to establish a priori L_1^2 bounds
- These L_1^2 bounds imply weak compactness (Hodge theory for abelian case, Uhlenbeck/Sedlacek theory for non-abelian case)
- Elliptic regularity implies strong (Uhlenbeck) compactness

Summary

Using established analytic machinery, *a priori L_1^2 bounds imply compactness*

Examples: ASD, Seiberg-Witten, PU(2) monopoles

The standard compactness strategy

- Use energy identities to establish a priori L_1^2 bounds
- These L_1^2 bounds imply weak compactness (Hodge theory for abelian case, Uhlenbeck/Sedlacek theory for non-abelian case)
- Elliptic regularity implies strong (Uhlenbeck) compactness

Summary

Using established analytic machinery, *a priori L_1^2 bounds imply compactness*

Examples: ASD, Seiberg-Witten, PU(2) monopoles

The standard compactness strategy

- Use energy identities to establish a priori L_1^2 bounds
- These L_1^2 bounds imply weak compactness (Hodge theory for abelian case, Uhlenbeck/Sedlacek theory for non-abelian case)
- Elliptic regularity implies strong (Uhlenbeck) compactness

Summary

Using established analytic machinery, *a priori L_1^2 bounds imply compactness*

Examples: ASD, Seiberg-Witten, PU(2) monopoles

The standard compactness strategy

- Use energy identities to establish a priori L_1^2 bounds
- These L_1^2 bounds imply weak compactness (Hodge theory for abelian case, Uhlenbeck/Sedlacek theory for non-abelian case)
- Elliptic regularity implies strong (Uhlenbeck) compactness

Summary

Using established analytic machinery, *a priori L_1^2 bounds imply compactness*

Examples: ASD, Seiberg-Witten, $\mathrm{PU}(2)$ monopoles

Energy identities for Vafa-Witten

We emulate the standard approach:

$$\mathcal{E}_{\mathrm{VW}}(C, A, B) := \tfrac{1}{2}\left\|d_A C + d_A^* B\right\|^2 + \left\|F_A^+ - \tfrac{1}{4}\left[B \times B\right] - \tfrac{1}{2}\left[C, B\right]\right\|^2$$

$$= \tfrac{1}{2}\left\|d_A C\right\|^2 + \tfrac{1}{2}\left\|d_A^* B\right\|^2 + \left\|F_A^+ - \tfrac{1}{4}\left[B \times B\right]\right\|^2 + \tfrac{1}{4}\left\|\left[C, B\right]\right\|^2$$

$$+ \int_X \left(\langle d_A C, d_A^* B\rangle - \langle F_A^+, [C, B]\rangle\right) + \int_X \tfrac{1}{4}\left\langle [B \times B], [C, B]\right\rangle.$$

The bottom line cancels since

$$\langle F_A^+, [C, B]\rangle = \langle [F_A^+, C], B\rangle = \langle d_A d_A C, B\rangle,$$

and the Jacobi identity implies

$$[[B \times B] \cdot B] = 0.$$

Energy identities for Vafa-Witten

We emulate the standard approach:

$$\mathcal{E}_{\mathrm{VW}}(C,A,B) := \tfrac{1}{2}\left\|d_A C + d_A^* B\right\|^2 + \left\|F_A^+ - \tfrac{1}{4}[B\times B] - \tfrac{1}{2}[C,B]\right\|^2$$

$$= \tfrac{1}{2}\left\|d_A C\right\|^2 + \tfrac{1}{2}\left\|d_A^* B\right\|^2 + \left\|F_A^+ - \tfrac{1}{4}[B\times B]\right\|^2 + \tfrac{1}{4}\left\|[C,B]\right\|^2$$
$$+ \int_X \left(\langle d_A C, d_A^* B\rangle - \langle F_A^+, [C,B]\rangle\right) + \int_X \tfrac{1}{4}\langle [B\times B], [C,B]\rangle .$$

The bottom line cancels since

$$\langle F_A^+, [C,B]\rangle = \langle [F_A^+, C], B\rangle = \langle d_A d_A C, B\rangle ,$$

and the Jacobi identity implies

$$[[B\times B]\cdot B] = 0.$$

Energy identities for Vafa-Witten

We emulate the standard approach:

$$\mathcal{E}_{\mathrm{VW}}(C, A, B) := \tfrac{1}{2}\left\|d_A C + d_A^* B\right\|^2 + \left\|F_A^+ - \tfrac{1}{4}[B \times B] - \tfrac{1}{2}[C, B]\right\|^2$$

$$= \tfrac{1}{2}\left\|d_A C\right\|^2 + \tfrac{1}{2}\left\|d_A^* B\right\|^2 + \left\|F_A^+ - \tfrac{1}{4}[B \times B]\right\|^2 + \tfrac{1}{4}\left\|[C, B]\right\|^2$$
$$+ \int_X \left(\langle d_A C, d_A^* B\rangle - \langle F_A^+, [C, B]\rangle\right) + \int_X \tfrac{1}{4}\langle [B \times B], [C, B]\rangle .$$

The bottom line cancels since

$$\langle F_A^+, [C, B]\rangle = \langle [F_A^+, C], B\rangle = \langle d_A d_A C, B\rangle ,$$

and the Jacobi identity implies

$$[[B \times B] \cdot B] = 0.$$

Simplification

Thus, assuming that the base manifold X is closed, we have the identity

$$\mathcal{E}_{\mathrm{VW}} = \frac{1}{2}\left\|d_A C\right\|^2 + \frac{1}{2}\left\|d_A^* B\right\|^2 + \left\|F_A^+ - \frac{1}{4}\left[B \times B\right]\right\|^2 + \frac{1}{4}\left\|\left[C, B\right]\right\|^2.$$

This is a different sum of squares, equivalent equations are

$$F_A^+ - \frac{1}{4}\left[B \times B\right] = 0, \qquad \left[C, B\right] = 0,$$
$$d_A^* B = 0, \qquad d_A C = 0.$$

These equations are linear in C. The interesting nonlinear part with B decouples. WLOG, set $C = 0$.

Simplification

Thus, assuming that the base manifold X is closed, we have the identity

$$\mathcal{E}_{\mathrm{VW}} = \tfrac{1}{2}\left\|d_A C\right\|^2 + \tfrac{1}{2}\left\|d_A^* B\right\|^2 + \left\|F_A^+ - \tfrac{1}{4}\left[B \times B\right]\right\|^2 + \tfrac{1}{4}\left\|\left[C, B\right]\right\|^2.$$

This is a different sum of squares, equivalent equations are

$$F_A^+ - \tfrac{1}{4}\left[B \times B\right] = 0, \qquad \left[C, B\right] = 0,$$
$$d_A^* B = 0, \qquad d_A C = 0.$$

These equations are linear in C. The interesting nonlinear part with B decouples. WLOG, set $C = 0$.

Analogy with Seiberg-Witten

The equations

$$F_A^+ = \tfrac{1}{4}\left[B \times B\right]$$
$$d_A^* B = 0$$

These equations say that B has a harmonic square root, if we interpret "B^2" $= \left[B \times B\right]$.

$$B = \text{“}2\sqrt{F_A^+}\text{”}$$
$$d_A^* B = 0 \quad (\Rightarrow d_A B = 0)$$

Contrast this with the Seiberg-Witten equations

$$F_A^+ - (\phi \otimes \phi^*)_0 = 0$$
$$\partial\!\!\!/_A \phi = 0$$

Analogy with Seiberg-Witten

The equations

$$F_A^+ = \tfrac{1}{4}\left[B \times B\right]$$
$$d_A^* B = 0$$

These equations say that B has a harmonic square root, if we interpret "B^2" $= \left[B \times B\right]$.

$$B = \text{“}2\sqrt{F_A^+}\text{”}$$
$$d_A^* B = 0 \quad (\Rightarrow d_A B = 0)$$

Contrast this with the Seiberg-Witten equations

$$F_A^+ - (\phi \otimes \phi^*)_0 = 0$$
$$\not{\partial}_A \phi = 0$$

The Weitzenböck formula

$$\tfrac{1}{2}\left\|d_A^* B\right\|^2 = \tfrac{1}{4}\left\|\nabla_A B\right\|^2 + \int_X \Big(\tfrac{1}{2}\left\langle B, [F_A^+ \times B]\right\rangle + \\ + \left(\tfrac{1}{12}s - \tfrac{1}{2}W^+\right)\cdot\left\langle B \otimes B\right\rangle\Big).$$

$$\mathcal{E}_{\mathrm{VW}}(0, A, B) = \left\|F_A^+ - \tfrac{1}{4}[B \times B]\right\|^2 + \tfrac{1}{2}\left\|d_A^* B\right\|^2 .$$

Once again, the cross-term miraculously cancels:

$$\mathcal{E}_{\mathrm{VW}} = \left\|F_A^+\right\|^2 + \tfrac{1}{16}\left\|[B \times B]\right\|^2 + \tfrac{1}{4}\left\|\nabla_A B\right\|^2 + \\ \int_X \tfrac{1}{2}\Big(\left\langle B, [F_A^+ \times B]\right\rangle - \left\langle F_A^+, [B \times B]\right\rangle\Big) + \int_X \left(\tfrac{1}{12}s - \tfrac{1}{2}W^+\right)\cdot\left\langle B \otimes B\right\rangle .$$

The Weitzenböck formula

$$\tfrac{1}{2}\left\|d_A^* B\right\|^2 = \tfrac{1}{4}\left\|\nabla_A B\right\|^2 + \int_X \Big(\tfrac{1}{2}\left\langle B, [F_A^+ \times B]\right\rangle + \\ + \left(\tfrac{1}{12}s - \tfrac{1}{2}W^+\right)\cdot\left\langle B \otimes B\right\rangle \Big).$$

$$\mathcal{E}_{\mathrm{VW}}(0, A, B) = \left\|F_A^+ - \tfrac{1}{4}[B \times B]\right\|^2 + \tfrac{1}{2}\left\|d_A^* B\right\|^2 .$$

Once again, the cross-term miraculously cancels:

$$\mathcal{E}_{\mathrm{VW}} = \left\|F_A^+\right\|^2 + \tfrac{1}{16}\left\|[B \times B]\right\|^2 + \tfrac{1}{4}\left\|\nabla_A B\right\|^2 + \\ \int_X \tfrac{1}{2}\Big(\cancel{\left\langle B, [F_A^+ \times B]\right\rangle - \left\langle F_A^+, [B \times B]\right\rangle}\Big) + \int_X \left(\tfrac{1}{12}s - \tfrac{1}{2}W^+\right)\cdot\left\langle B \otimes B\right\rangle .$$

Vanishing theorem

The Vafa-Witten equations (with $C = 0$) are equivalent to

$$0 = \mathcal{E}_{\mathrm{VW}} = \tfrac{1}{2}\left\|F_A^+\right\|^2 + \tfrac{1}{4}\left\|\nabla_A B\right\|^2 + \tfrac{1}{16}\left\|[B \times B]\right\|^2 + \\ + \int_X \left(\tfrac{1}{12}s - \tfrac{1}{2}W^+\right) \cdot \langle B \otimes B\rangle\,.$$

If furthermore the curvature part is positive semi-definite, then M must be Kähler, hyper-Kähler, or $b^+ = 0$, and the equations decouple further to

$$F_A^+ = 0 \qquad \nabla_A B = 0 \qquad [B \times B] = 0.$$

Vanishing theorem

The Vafa-Witten equations (with $C = 0$) are equivalent to

$$0 = \mathcal{E}_{\mathrm{VW}} = \tfrac{1}{2} \left\| F_A^+ \right\|^2 + \tfrac{1}{4} \left\| \nabla_A B \right\|^2 + \tfrac{1}{16} \left\| [B \times B] \right\|^2 + + \int_X \left(\tfrac{1}{12} s - \tfrac{1}{2} W^+ \right) \cdot \langle B \otimes B \rangle .$$

If furthermore the curvature part is positive semi-definite, then M must be Kähler, hyper-Kähler, or $b^+ = 0$, and the equations decouple further to

$$F_A^+ = 0 \qquad \nabla_A B = 0 \qquad [B \times B] = 0.$$

Failure of a priori bound

Recall that our quartic term $\|[B \,\boldsymbol{.}\, B]\|^2$ is only positive *semi*-definite. If it were positive-definite, then it would dominate the curvature part, and the identity

$$0 = \tfrac{1}{2}\left\|F_A\right\|^2 + \tfrac{1}{4}\left\|\nabla_A B\right\|^2 + \tfrac{1}{16}\left\|[B \times B]\right\|^2 + \\ + \int_X \left(\tfrac{1}{12}s - \tfrac{1}{2}W^+\right) \cdot \langle B \otimes B \rangle - \kappa$$

would yield a priori bounds on $\|F_A\|$, $\|\nabla_A B\|$, and $\|B\|_{L^4}$. Instead, we get no such bounds since $\|[B \times B]\|^2$ could vanish while the curvature terms go to $-\infty$ unchecked.

More Weitzenböck

For a more concrete application of the width heuristic, consider the following identity for solutions:

$$\tfrac{1}{8}\Delta |B|^2 + \tfrac{1}{4} |\nabla_A B|^2 + \tfrac{1}{8} |[B \times B]|^2 = \langle B \cdot (-\tfrac{1}{12}s + \tfrac{1}{2}W^+) B \rangle$$

In particular,

$$\Delta |B|^2 \leq \lambda |B|^2$$

where λ depends on curvature.

With slightly more work,

$$\Delta |B| \leq \lambda |B|$$

More Weitzenböck

For a more concrete application of the width heuristic, consider the following identity for solutions:

$$\tfrac{1}{8}\Delta |B|^2 + \tfrac{1}{4} |\nabla_A B|^2 + \tfrac{1}{8} |[B \times B]|^2 = \langle B \cdot (-\tfrac{1}{12}s + \tfrac{1}{2}W^+) B \rangle$$

In particular,

$$\Delta |B|^2 \leq \lambda |B|^2$$

where λ depends on curvature.

With slightly more work,

$$\Delta |B| \leq \lambda |B|$$

Bounding B

Thanks to a mean-value inequality due to Morrey

$$\Delta |B| \leq \lambda |B| \implies \|B\|_{L^\infty} \leq c \|B\|_{L^2}$$

Thus

$$\|F_A^+\|_{L^\infty} = \left\|\tfrac{1}{4}[B \times B]\right\|_{L^\infty} \leq c' \|B\|_{L^2}^2 .$$

Assuming a bound on $\|B\|_{L^2}$, we get bounds on $\|F_A^+\|_{L^\infty}$ and $\|B\|_{L^\infty}$.

If $|[B \times B]|^2$ were positive-definite, such bounds would follow automatically from a priori estimates plus maximum principle.

Bounding B

Thanks to a mean-value inequality due to Morrey

$$\Delta |B| \leq \lambda |B| \implies \|B\|_{L^\infty} \leq c \|B\|_{L^2}$$

Thus

$$\|F_A^+\|_{L^\infty} = \|\tfrac{1}{4}[B \times B]\|_{L^\infty} \leq c' \|B\|_{L^2}^2 .$$

Assuming a bound on $\|B\|_{L^2}$, we get bounds on $\|F_A^+\|_{L^\infty}$ and $\|B\|_{L^\infty}$.

If $|[B \times B]|^2$ were positive-definite, such bounds would follow automatically from a priori estimates plus maximum principle.

Bounding B

Thanks to a mean-value inequality due to Morrey

$$\Delta |B| \leq \lambda |B| \implies \|B\|_{L^\infty} \leq c \|B\|_{L^2}$$

Thus

$$\|F_A^+\|_{L^\infty} = \left\|\tfrac{1}{4}[B \times B]\right\|_{L^\infty} \leq c' \|B\|_{L^2}^2 .$$

Assuming a bound on $\|B\|_{L^2}$, we get bounds on $\|F_A^+\|_{L^\infty}$ and $\|B\|_{L^\infty}$.

If $|[B \times B]|^2$ were positive-definite, such bounds would follow automatically from a priori estimates plus maximum principle.

Bounding B

Thanks to a mean-value inequality due to Morrey

$$\Delta |B| \leq \lambda |B| \implies \|B\|_{L^\infty} \leq c \|B\|_{L^2}$$

Thus

$$\|F_A^+\|_{L^\infty} = \left\|\tfrac{1}{4}[B \times B]\right\|_{L^\infty} \leq c' \|B\|_{L^2}^2 .$$

Assuming a bound on $\|B\|_{L^2}$, we get bounds on $\|F_A^+\|_{L^\infty}$ and $\|B\|_{L^\infty}$.

If $|[B \times B]|^2$ were positive-definite, such bounds would follow automatically from a priori estimates plus maximum principle.

Feehan-Leness program for $\mathrm{PU}(2)$

Only major property distinguishing $\mathrm{PU}(2)$ monopoles and Vafa-Witten equations is $|[B \times B]|^2$ being semi-definite. Their analytic framework extends to give:

- Slice theorem
- Elliptic estimates
- Removal of singularities
- Compactness (almost!)

Compactness requires bounds on $\|F_A^+\|_{L^\infty}$ and $\|B\|_{L^\infty}$.

Feehan-Leness program for $\mathrm{PU}(2)$

Only major property distinguishing $\mathrm{PU}(2)$ monopoles and Vafa-Witten equations is $|[B \times B]|^2$ being semi-definite. Their analytic framework extends to give:

- Slice theorem
- Elliptic estimates
- Removal of singularities
- Compactness (almost!)

Compactness requires bounds on $\|F_A^+\|_{L^\infty}$ and $\|B\|_{L^\infty}$.

Truncated Vafa-Witten moduli space

$$\mathcal{M}^b_{\mathrm{VW},k} := \left\{ [0, A, B] \in \mathcal{M}_{\mathrm{VW},k} \;\middle|\; \|B\|_{L^2} \le b \right\}, \;\; b \in \mathbb{R}$$

- $\mathcal{M}^b_{\mathrm{VW},k} \subset \mathcal{M}^{b'}_{\mathrm{VW},k}$ for $b \le b'$.
- $\mathcal{M}^0_{\mathrm{VW},k} = \mathcal{M}_{\mathrm{ASD},k}$.
- $\mathcal{M}^b_{\mathrm{VW},k} = \varnothing$ for $b < 0$ or $k < -cb^4$

Each $\mathcal{M}^b_{\mathrm{VW},k}$ has an Uhlenbeck compactification $\overline{\mathcal{M}}^b_{\mathrm{VW},k}$.

A partial compactification is given by $\overline{\mathcal{M}}_{\mathrm{VW},k} := \bigcup_{b \in \mathbb{R}} \overline{\mathcal{M}}^b_{\mathrm{VW},k}$.

Truncated Vafa-Witten moduli space

$$\mathcal{M}^{b}_{\mathrm{VW},k} := \left\{ [0, A, B] \in \mathcal{M}_{\mathrm{VW},k} \;\middle|\; \|B\|_{L^2} \leq b \right\}, \;\; b \in \mathbb{R}$$

- $\mathcal{M}^{b}_{\mathrm{VW},k} \subset \mathcal{M}^{b'}_{\mathrm{VW},k}$ for $b \leq b'$.
- $\mathcal{M}^{0}_{\mathrm{VW},k} = \mathcal{M}_{\mathrm{ASD},k}$.
- $\mathcal{M}^{b}_{\mathrm{VW},k} = \varnothing$ for $b < 0$ or $k < -cb^4$

Each $\mathcal{M}^{b}_{\mathrm{VW},k}$ has an Uhlenbeck compactification $\overline{\mathcal{M}}^{b}_{\mathrm{VW},k}$.

A partial compactification is given by $\overline{\mathcal{M}}_{\mathrm{VW},k} := \bigcup_{b \in \mathbb{R}} \overline{\mathcal{M}}^{b}_{\mathrm{VW},k}$.

Truncated Vafa-Witten moduli space

$$\mathcal{M}^b_{\mathrm{VW},k} := \left\{ [0, A, B] \in \mathcal{M}_{\mathrm{VW},k} \;\middle|\; \|B\|_{L^2} \le b \right\}, \quad b \in \mathbb{R}$$

- $\mathcal{M}^b_{\mathrm{VW},k} \subset \mathcal{M}^{b'}_{\mathrm{VW},k}$ for $b \le b'$.
- $\mathcal{M}^0_{\mathrm{VW},k} = \mathcal{M}_{\mathrm{ASD},k}$.
- $\mathcal{M}^b_{\mathrm{VW},k} = \varnothing$ for $b < 0$ or $k < -cb^4$

Each $\mathcal{M}^b_{\mathrm{VW},k}$ has an Uhlenbeck compactification $\overline{\mathcal{M}}^b_{\mathrm{VW},k}$.

A partial compactification is given by $\overline{\mathcal{M}}_{\mathrm{VW},k} := \bigcup_{b \in \mathbb{R}} \overline{\mathcal{M}}^b_{\mathrm{VW},k}$.

IT'S OVER!

- Thanks for listening!!