Some Analytic Aspects of Vafa-Witten Twisted $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

by

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Abstract

Given an oriented Riemannian four-manifold equipped with a principal bundle, we investigate the moduli space $\mathcal{M}_{\mathrm{VW}}$ of solutions to the Vafa-Witten equations. These equations arise from a twist of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Physicists believe that this theory has a well-defined partition function, which depends on a single complex parameter. On one hand, the S-duality conjecture predicts that this partition function is a modular form. On the other hand, the Fourier coefficients of the partition function are supposed to be the "Euler characteristics" of various moduli spaces $\overline{\mathcal{M}}_{\mathrm{ASD}}$ of compactified anti-self-dual instantons. For several algebraic surfaces, these Euler characteristics were verified to be modular forms.

Except in certain special cases, it's unclear how to precisely define the partition function. If there is a mathematically sensible definition of the partition function, we expect it to arise as a gauge-theoretic invariant of the moduli spaces $\mathcal{M}_{\mathrm{VW}}$. The aim of this thesis is to initiate the analysis necessary to define such invariants. We establish various properties, computations, and estimates for the Vafa-Witten equations. In particular, we give a partial Uhlenbeck compactification of the moduli space.

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Contents

Chapter 1

Introduction to twisted $\mathcal{N} = 4$ supersymmetric instantons

1.1 Yang-Mills theory and the Donaldson invariants

During the 1980s and the early 1990s, the moduli space of anti-self-dual (ASD) instantons was a crucial tool for the study of smooth four-manifolds. Let (X, g) be an oriented Riemannian four-manifold, and let G be a compact Lie group. Given a principal G-bundle $P \to X$, any connection $A \in \mathcal{A}_P$, has a curvature $F_A \in \Omega^2(X; \mathfrak{g}_P)$. Two-forms split into metric-dependent ± 1 eigenspaces $\Omega^{2,\pm}(X; \mathfrak{g}_P)$ of the Hodge star operator. Correspondingly, curvature decomposes as $F_A = F_A^+ + F_A^-$. For a specific choice of (X, g) and P, we define *ASD moduli space* as

$$\mathcal{M}_{\mathrm{ASD}}(P, g) := \{A \in \mathcal{A}_P \mid F_A^+ = 0\} / \mathcal{G}_P,$$

where $\mathcal{G}_P$ is the group of automorphisms of P. When non-empty, $\mathcal{M}_{\mathrm{ASD}}$ typically is a submanifold of finite dimension inside the quotient $\mathcal{A}_P/\mathcal{G}_P$. In the case $G = \mathrm{SU}(2)$, Donaldson showed how the homology of $\mathcal{M}_{\mathrm{ASD}} \subset \mathcal{A}_P/\mathcal{G}_P$ defines (in many circumstances) invariants of 4-manifolds which are independent of the metric g, and capable of distinguishing differentiable structures [Don90, DK97].

Proving that the Donaldson invariants are well-defined involves many technical challenges. For example, sometimes $\mathcal{M}_{\mathrm{ASD}}$ is usually non-compact, and sometimes has singularities. There is a natural Uhlenbeck compactification $\overline{\mathcal{M}}_{\mathrm{ASD}}$, and singularities are handled through metric perturbations. Furthermore, while the topology of $\overline{\mathcal{M}}_{\mathrm{ASD}}(P, g)$ depends on g, one must prove that the invariants do not. Roughly, this amounts to showing that different choices of metric lead to homologous $\overline{\mathcal{M}}_{\mathrm{ASD}}$.

1.2 A brief overview of twisted supersymmetric Yang-Mills theory

Supersymmetry provides much of the historical motivation behind the questions this thesis attempts to address. Because supersymmetry does not play an essential role in this thesis, this overview will be cursory.

Roughly speaking, supersymmetry relates fermionic (odd / antisymmetric) particles to bosonic (even / symmetric) particles. A supersymmetic theory comes equipped with $\mathcal{N}$ supersymmetry operators $\{Q_i\}_{i=1}^{\mathcal{N}}$ which exchange bosons and fermions. Analysis of twisted $\mathcal{N}=2$ supersymmetric Yang-Mills theory [SW94a, SW94b] led to the discovery of the Seiberg-Witten equations [Wit94].

There are three possible twists of the $\mathcal{N}=4$ supersymmetric Yang-Mills equations. The focus of this thesis is the *Vafa-Witten twist,* which was studied in [VW94]. Recently, a different twist was shown to be related to the geometric Langlands program and dubbed the *GL twist* [KW07].

Supersymmetric quantum field theories have nice properties which make them relatively tractable, and physicists conjectured an electric-magnetic duality which exchanges strong and weak coupling when $\mathcal{N}=4$. This is known as *S-duality*.

1.3 The Vafa-Witten invariant

In search of evidence for S-duality, Vafa and Witten explored their twist of $\mathcal{N}=4$ supersymmetric Yang-Mills theory [VW94]. For any smooth oriented Riemannian four-manifold (X, g_0) with principal bundle $P \to X$, we define the configuration spaces

$$\begin{aligned} \mathcal{C} &:= \Omega^0(X;\mathfrak{g}_P) \times \mathcal{A}_P \times \Omega^{2,+}(X;\mathfrak{g}_P), \\ \mathcal{C}' &:= \Omega^1(X;\mathfrak{g}_P) \times \Omega^{2,+}(X;\mathfrak{g}_P). \end{aligned}$$

We define the gauge-equivariant *Vafa-Witten map*[1]

$$\begin{aligned} \mathrm{VW} &: \mathcal{C} \longrightarrow \mathcal{C}', \\ \mathrm{VW}(C,A,B) &:= \begin{pmatrix} (d_A C + d_A^* B)/\sqrt{2} \\ F_A^+ + \frac{1}{8}[B \centerdot B] + \frac{1}{2}[B,C] \end{pmatrix}. \end{aligned} \tag{1.1}$$

Mimicking the setup of Donaldson theory, the *Vafa-Witten moduli space* is

$$\mathcal{M}_{\mathrm{VW}}(P,g) := \{(C,A,B) \mid \mathrm{VW}(C,A,B)=0\}/\mathcal{G}_P.$$

When the map VW is transverse, the expected dimension of $\mathcal{M}_{\mathrm{VW}}$ is zero since the principal symbol of (1.1) (plus gauge fixing) is self-adjoint. However, transversality often fails, resulting in components of positive dimension. In particular $\mathcal{M}_{\mathrm{ASD}} \subset \mathcal{M}_{\mathrm{VW}}$ since

$$\mathrm{VW}(0,A,0) = \begin{pmatrix} 0 \\ F_A^+ \end{pmatrix}.$$

By appropriately counting the number of points/components of $\mathcal{M}_{\mathrm{VW}}$, we hope to obtain a (conjecturally well-defined) number $\mathrm{VW}(P)$ called the *Vafa-Witten invariant* for the principal bundle $P \to X$. Vafa and Witten argue that the number $\mathrm{VW}(P)$ corresponds to the formal Atiyah-Jeffrey

[1] The factor $1/\sqrt{2}$ is chosen to ensure that (2.1) works out. The product denoted by $\beta_1 \centerdot \beta_2$ for $\beta_i \in \Omega^{2,+}(X;\mathbb{R})$ is fiberwise-equivalent to a multiple of the cross product $\times$ on $\mathbb{R}^3$. The notation $[B \centerdot B]$ is described in Sections A.1.6 and A.1.7. The correspondence between the (1.1) and the expression given in [VW94] is described in Section 2.3.

expression [AJ90] for the Euler characteristic of $\mathcal{M}_{\mathrm{ASD}}(P, g)$. Although this suggests a geometric interpretation of $\mathrm{VW}(P)$, the formal Atiyah-Jeffrey expression is a mathematically dubious infinite-dimensional integral. It's very unclear to what extent $\mathrm{VW}(P)$ is well-defined, and the technical challenges involved are at least an order of magnitude greater than those of the Donaldson invariants.

1.3.1 Euler characteristics of instanton moduli space

Many instances of the S-duality conjecture have been verified by assuming that $B = 0$, using "the Euler characteristic of $\overline{\mathcal{M}}_{\mathrm{ASD}}$" as a working definition of the Vafa-Witten invariant, and then computing $\chi(\overline{\mathcal{M}}_{\mathrm{ASD}})$ via algebro-geometric methods.

For example, when $X = \mathrm{K3}$, using partial results about the Euler characteristics of $\overline{\mathcal{M}}_{\mathrm{ASD}}$, Vafa and Witten proposed [VW94, §4.1] that the partition functions for $G = \mathrm{SU}(2)$ and $\hat{G} = \mathrm{SO}(3)$ should be

$$\begin{aligned} Z_{\mathrm{SU}(2)}(q) &= \tfrac{1}{2}q^{-2}\left(\tfrac{1}{4} + 0q + 30q^2 + 3200q^3 + \cdots + \tfrac{1018979075 6178504975}{4}q^{16} + \cdots\right. \\ Z_{\mathrm{SO}(3)}(q) &= q^{-2}\left(\tfrac{1}{4} + 0q^{1/2} + 0q + 2096128q^{3/2} + 50356230q^2 + 679145472q^{5/2} + \cdots\right. \\ &\qquad \cdots + \tfrac{2137997440957227092282 4975}{4}q^{16} + \cdots \end{aligned}$$

Here, the generating functions for the Euler characteristics are inside the parentheses, and differ from the partition function by an overall factor of $\frac{1}{2}q^{-2}$ or q^{-2}, respectively. The denominators apparently result from orbifold singularities in certain $\overline{\mathcal{M}}_{\mathrm{ASD}}$.

If we define the parameter τ in the upper-half-plane by the relation $q^{1/2} = e^{i\pi\tau}$, then $Z_{\mathrm{SU}(2)}$ and $Z_{\mathrm{SO}(3)}$ are periodic functions of τ, and the q-series are Fourier series. The groups $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ are Langlands-dual to each other, and the S-duality conjecture relates their partition functions. In this case, they obey the "modular relation"

$$Z_{\mathrm{SU}(2)}(-1/\tau) = (2\tau)^{-12} Z_{\mathrm{SO}(3)}(\tau).$$

Note: this is especially remarkable since if $Z(\tau)$ is an arbitrary periodic function, then there's no reason to expect that $(2\tau)^{12} Z(-1/\tau)$ is also periodic.

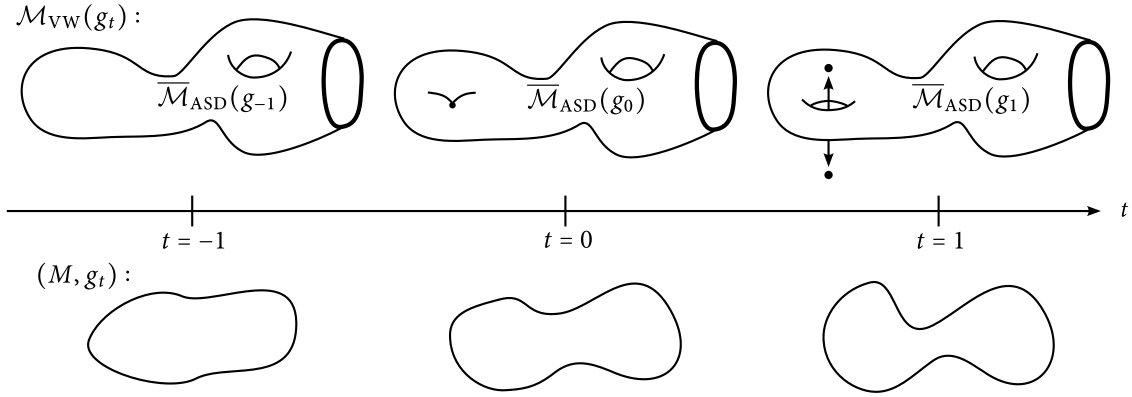
For more examples of these sorts of formulas, see the recent article [Wu08], and the references therein.

Despite this success, many troubling technical issues remain to be addressed. For example,

1. When is it safe to assume $B = 0$?
2. How should singularities of $\overline{\mathcal{M}}_{\mathrm{ASD}}$ be counted?
3. To what extent is the Euler characteristic independent of the metric?
4. How does the choice of compactification affect the Euler characteristic?

The first question of the vanishing of B is partially addressed in Remark 2.1.4, with further results for the Kähler case in [VW94, §2.4].

Figure 1-1: An idealized view of $\mathcal{M}_{\mathrm{VW}}$: while deforming the metric g_t on M, the Euler characteristic of $\overline{\mathcal{M}}_{\mathrm{ASD}}$ can change. Any change to $\chi(\overline{\mathcal{M}}_{\mathrm{ASD}})$ should balance with the creation or destruction of points in $\mathcal{M}_{\mathrm{VW}} \backslash \mathcal{M}_{\mathrm{ASD}}$.



For the second question of singularities, one might employ a suitable transversality theorem, such as the generic metrics theorem for ASD connections. Then one must check that the invariant either does not depend on the perturbation, or find a suitable wall-crossing formula.

Third, as the metric changes, the topology of $\overline{\mathcal{M}}_{\mathrm{ASD}}$ can change, and $\chi(\overline{\mathcal{M}}_{\mathrm{ASD}})$ is not invariant. However, we should expect some compensation in the full moduli space $\overline{\mathcal{M}}_{\mathrm{VW}}$, as in Figure 1-1 on page 14. In this thesis, we began to work out models for these changes in topology in Chapter 5, and on the Uhlenbeck boundary in Chapter 9.

Finally, we expect to better understand the choice of compactification when we complete the program of Chapter 9. Other work by Li and Qin contrasts the Uhlenbeck and Gieseker compactifications in their study of blowup formulae [LQ99, LQ98, LQ02].

1.3.2 Formal similarity to Seiberg-Witten

We will be primarily interested in the case $C = 0$, for which the equations reduce to

$$\begin{aligned} d_A^* B &= 0, \\ F_A^+ + \tfrac{1}{8}[B \centerdot B] &= 0. \end{aligned}$$

Compare this to the Seiberg-Witten equations for a $\mathrm{U}(1)$ principal bundle P' and a pair $(A', \Psi) \in \mathcal{A}_{P'} \times \Omega^0(X; \not{S}^+)$

$$\begin{aligned} \not{\partial}_{A'} \Psi &= 0, \\ F_{A'}^+ - \rho^{-1}(\Psi \otimes \Psi^*)_0 &= 0. \end{aligned}$$

Both sets of equations can be regarded as the condition that self-dual curvature has a harmonic "square root" in respective representations. Specifically, B is a square root of F_A^+ with respect to the quadratic map $-\frac{1}{8}[B \centerdot B]$, while Ψ is a square root of $F_{A'}^+$ with respect to the quadratic map $\rho^{-1}(\Psi \otimes \Psi^*)_0$.

Chapter 2

Fundamental energy identities

2.1 Energy identities

Let (X, g_0) be an oriented Riemannian four-manifold with boundary, equipped with a principal bundle $P \to X$. We define the *gradient energy density*, which is the measure given by

$$\mu_{\text{grad}}(C, A, B) := |\mathrm{VW}(C, A, B)|^2 \, d\text{vol} = \left(\tfrac{1}{2} |d_A C + d_A^* B|^2 + |F_A^+ + \tfrac{1}{8} [B \centerdot B] + \tfrac{1}{2} [B, C]|^2 \right) d\text{vol}. \quad (2.1)$$

Correspondingly, we define the gradient energy

$$\varepsilon_{\text{grad}}(C, A, B) := \int_X \mu_{\text{grad}}(C, A, B) = \tfrac{1}{2} \|d_A C + d_A^* B\|^2 + \|F_A^+ + \tfrac{1}{8} [B \centerdot B] + \tfrac{1}{2} [B, C]\|^2, \quad (2.2)$$

so that solutions to the Vafa-Witten equations are precisely those configurations for which $\varepsilon_{\text{grad}} = 0$. We gain tremendous insight by rewriting $\varepsilon_{\text{grad}}$, integrating by parts and applying the Weitzenböck formula. Several of these ideas are also explained in [VW94] §2.4.

We begin by expanding a few terms:

$$\begin{aligned} \varepsilon_{\text{grad}} = \tfrac{1}{2} \|d_A C\|^2 + \tfrac{1}{2} \|d_A^* B\|^2 + \int_X \langle d_A C \cdot d_A^* B \rangle + \\ + \|F_A^+ + \tfrac{1}{8} [B \centerdot B]\|^2 + \tfrac{1}{4} \|[B, C]\|^2 + \tfrac{1}{8} \int_X \langle [B \centerdot B] \cdot [B, C] \rangle + \int_X \langle F_A^+ \cdot [B, C] \rangle . \end{aligned}$$

To deal with the cross terms, note that

$$\langle F_A^+ \wedge \star [B, C] \rangle + \langle d_A C \wedge \star d_A^* B \rangle = -d \langle d_A C \wedge B \rangle ,$$

and

$$\langle [B \centerdot B] \cdot [B, C] \rangle = \langle [[B \centerdot B] \cdot B], C \rangle = 0,$$

since $[[B \centerdot B] \cdot B]$ vanishes by the Jacobi identity (A.24). Thus

$$\varepsilon_{\text{grad}} = \tfrac{1}{2} \|d_A C\|^2 + \tfrac{1}{2} \|d_A^* B\|^2 + \|F_A^+ + \tfrac{1}{8} [B \centerdot B]\|^2 + \tfrac{1}{4} \|[B, C]\|^2 - \int_{\partial X} \left\langle d_{A^\parallel} C^\parallel \cdot \star B^\parallel \right\rangle ,$$

where $A^{\|}, B^{\|}, C^{\|}$ are the pullbacks of A, B, C to ∂X. This proves:

Theorem 2.1.1. *If X is a closed oriented four-manifold equipped with a principal bundle $P \to X$, then every solution (C, A, B) to the equation $\mathrm{VW}(C, A, B) = 0$ satisfies*

$$\begin{aligned} F_A^+ + \tfrac{1}{8}[B \centerdot B] &= 0, \\ d_A C = d_A^* B &= 0, \\ [B, C] &= 0. \end{aligned} \tag{2.3}$$

Remark 2.1.2. For any fixed A and B, the equations (2.3) are linear in C. In particular, if A is an irreducible $\mathrm{SU}(2)$ connection, then the kernel of

$$d_A : \Omega^0(X; \mathfrak{g}_P) \longrightarrow \Omega^1(X; \mathfrak{g}_P)$$

is $\{0\}$, so $C = 0$.

We proceed with our manipulation of $\varepsilon_{\mathrm{grad}}$ by expanding the remaining term and using the Weitzenböck formula (B.17) to obtain

$$\begin{aligned} \varepsilon_{\mathrm{grad}} = \tfrac{1}{2}\|d_A C\|^2 + \|F_A^+\|^2 + \tfrac{1}{64}\|[B \centerdot B]\|^2 + \tfrac{1}{4}\|[B, C]\|^2 + \tfrac{1}{4}\|\nabla_A B\|^2 + \\ + \tfrac{1}{12}\int_X \left(s|B|^2 - 6W^+ \cdot \langle B \odot B\rangle\right) - \int_{\partial X}\left\langle d_{A^{\|}} C^{\|} \cdot \star B^{\|}\right\rangle - \tfrac{1}{2}\int_{\partial X}\left\langle B^{\|} \cdot (d_{A^{\|}} \star + N)\, B^{\|}\right\rangle. \end{aligned} \tag{2.4}$$

A simple consequence is:

Theorem 2.1.3. *If X is a closed manifold such that the quadratic form (see Section B.4)*

$$s - 6W^+ \in \Omega^0(X; \mathrm{Sym}^2(\Lambda^{2,+} T^* X))$$

is everywhere positive semi-definite, then every solution (C, A, B) satisfies

$$\begin{aligned} F_A^+ &= 0, & [B \centerdot B] &= 0, \\ \nabla_A B &= 0, & [B, C] &= 0, \\ d_A C &= 0. \end{aligned}$$

Remark 2.1.4. The condition $s - 6W^+ \geq 0$ is highly restrictive. According to the Weitzenböck formula for $\Omega^{2,+}(X; \mathbb{R})$, this implies that every self-dual two-form is covariantly constant. Since any nonzero covariantly constant self-dual two-form determines a Kähler structure on X, there are three possibilities for the Betti number b_2^+ when this condition could occur:

- $b_2^+ = 0$.
- $b_2^+ = 1$ so X is Kähler.
- $b_2^+ = 3$ so X is hyperKähler.

Turning now towards obtaining a priori estimates using $\varepsilon_{\text{grad}}$, we make one final substitution

$$\left\|F_A^+\right\|^2 = \tfrac{1}{2}\left\|F_A\right\|^2 + \tfrac{1}{2}\int_X \langle F_A \wedge F_A\rangle\,. \tag{2.5}$$

to obtain

$$\begin{aligned}\varepsilon_{\text{grad}} = \tfrac{1}{2}\left\|F_A\right\|^2 + \tfrac{1}{4}\left\|\nabla_A B\right\|^2 + \tfrac{1}{2}\left\|d_A C\right\|^2 + \tfrac{1}{4}\left\|[B,C]\right\|^2 + \tfrac{1}{64}\left\|[B\,\mathbf{.}\,B]\right\|^2 + \tfrac{1}{12}\int_X \left(s\,|B|^2 - 6W^+\cdot\langle B\odot B\rangle\right) + \\ + \tfrac{1}{2}\int_X \langle F_A\wedge F_A\rangle - \int_{\partial X}\left\langle d_{A^\parallel}C^\parallel \cdot \star B^\parallel\right\rangle - \tfrac{1}{2}\int_{\partial X}\left\langle B^\parallel\cdot\left(d_{A^\parallel}\star + N\right)B^\parallel\right\rangle.\end{aligned}$$

In analogy with [KM07] Definition 4.5.4, we define

$$\begin{aligned}\varepsilon_{\text{top}} &:= \int_{\partial X}\left\langle d_A C^\parallel\cdot\star B^\parallel\right\rangle + \tfrac{1}{2}\int_{\partial X}\left\langle B^\parallel\cdot\left(d_A\star + \rho_{\mathrm{D}}(N)\right)B^\parallel\right\rangle - \tfrac{1}{2}\int_X\langle F_A\wedge F_A\rangle\,,\\ \mu_{\text{an}} &:= \left(\tfrac{1}{2}|F_A|^2 + \tfrac{1}{4}|\nabla_A B|^2 + \tfrac{1}{2}|d_A C|^2 + \tfrac{1}{4}|[B,C]|^2 + \tfrac{1}{64}|[B\,\mathbf{.}\,B]|^2 + \tfrac{1}{12}s\,|B|^2 - \tfrac{1}{2}W^+\cdot\langle B\odot B\rangle\right)d\mathrm{vol},\\ \varepsilon_{\text{an}} &:= \int_X \mu_{\text{an}}.\end{aligned}$$

Note that ε_{top} depends only on the boundary values and the topological type of the principal bundle. These quantities have the following significance. Since $\varepsilon_{\text{grad}} = \varepsilon_{\text{an}} - \varepsilon_{\text{top}}$, we have the inequality

$$\varepsilon_{\text{an}} \geq \varepsilon_{\text{top}},$$

with equality holding precisely for solutions. In particular, ε_{top} is an a priori upper bound on ε_{an} for solutions. Ideally, we would like to use this to obtain an upper bound for $\|F_A\|$. For simplicity, we set $C = 0$ to obtain

$$\left\|F_A\right\|^2 = 2\varepsilon_{\text{top}} + \int_X\left(W^+\cdot\langle B\odot B\rangle - \tfrac{1}{6}s\,|B|^2\right) - \tfrac{1}{2}\left\|\nabla_A B\right\|^2 - \tfrac{1}{32}\left\|[B\,\mathbf{.}\,B]\right\|^2. \tag{2.6}$$

In the ASD case where X is closed and $B = 0$, we get the standard topological bound[1] on $2\varepsilon_{\text{top}}$ by a multiple of the "instanton number." For both the Seiberg-Witten equations [KM07] and the PU(2) monopole equations [Tel00, FL98] where this topological term alone does not suffice, the standard trick is to use the quartic term. If the quartic term $\frac{1}{32}\left\|[B\,\mathbf{.}\,B]\right\|^2$ were positive definite, it would dominate the Riemannian curvature terms. Unfortunately this is not the case, since the quartic term vanishes whenever B has rank one. Instead, the situation is analogous to Hitchin's equations for Higgs pairs [Hit87], where a priori L^2 bounds fail.

2.1.1 The Vafa-Witten Chern-Simons functional

In the spirit of [KM07] §4.3, we show that when X is a metric cylinder $\mathbb{R}\times Y$ the Vafa-Witten equations over X are the gradient flow of a functional over Y. Define

$$\mathrm{VWCS}(A^\parallel, B^\parallel, C^\parallel) := \tfrac{1}{2}\int_Y \mathrm{CS}(A^\parallel) + \int_Y\left\langle d_{A^\parallel}C^\parallel\wedge B^\parallel\right\rangle + \tfrac{1}{2}\int_Y B^\parallel\wedge d_A^\star B^\parallel,$$

[1] In this case, our bound is actually an equality: $2\varepsilon_{\text{top}} = (2\pi)^2 k$ from (C.1).

where $\mathrm{CS}(A^{\|})$ is given by $\mathrm{CS}(A^{\|}, A_0)$ of (C.2), for any fixed connection A_0.

We compute

$$
\begin{aligned}
\frac{\delta \mathrm{VWCS}}{\delta A^{\|}} &= -\star F_{A^{\|}} - \left[\star B^{\|}, C^{\|}\right] + \tfrac{1}{2} \star \left[\star B^{\|} \wedge \star B^{\|}\right], \\
\frac{\delta \mathrm{VWCS}}{\delta B^{\|}} &= d_A \star B^{\|} + \star d_A C^{\|}, \\
\frac{\delta \mathrm{VWCS}}{\delta C^{\|}} &= -\star d_A B^{\|}.
\end{aligned}
$$

Let A be a connection over X in temporal gauge, $B \in \Omega^{2,+}(X; \mathrm{ad}_P)$, and $C \in \Omega^0(X; \mathrm{ad}_P)$. The self-duality condition for B is $B^{\perp} = \star B^{\|}$. Some consequences of (B.10) include

$$
\begin{aligned}
F_A &= F_{A^{\|}} + dt \wedge (\dot{A}^{\|} - N A^{\|}), \\
d_A^* B &= \star \dot{B}^{\|} + \star d_A \star B^{\|} + N \star B^{\|} + dt \wedge \star d_A B^{\|}, \\
d_A C &= d_A C^{\|} + dt \wedge \dot{C}.
\end{aligned}
\tag{2.7}
$$

In each of these components, the gradient flow terms for the variables A, B, and C are

$$
\begin{aligned}
\dot{A}^{\|} - \frac{\delta \mathrm{VWCS}}{\delta A^{\|}} &= \star \left(2F_A^+ + \tfrac{1}{4}[B \,\text{.}\, B] + [B, C]\right)^{\|}, \\
\dot{B}^{\|} - \frac{\delta \mathrm{VWCS}}{\delta B^{\|}} &= -\star \left(d_A^* B + d_A C\right)^{\|}, \\
\dot{C}^{\|} - \frac{\delta \mathrm{VWCS}}{\delta C^{\|}} &= \left(d_A^* B + d_A C\right)^{\perp}.
\end{aligned}
$$

Therefore, the gradient flow for VWCS is given by the zeroes of

$$
\begin{aligned}
&\frac{1}{2}\left\|\dot{A}^{\|} - \frac{\delta \mathrm{VWCS}}{\delta A^{\|}}\right\|^2 + \frac{1}{2}\left\|\dot{B}^{\|} - \frac{\delta \mathrm{VWCS}}{\delta B^{\|}}\right\|^2 + \frac{1}{2}\left\|\dot{C}^{\|} - \frac{\delta \mathrm{VWCS}}{\delta C^{\|}}\right\|^2 \\
&= \tfrac{1}{2}\left\|d_A C + d_A^* B\right\|^2 + \left\|F_A^+ + \tfrac{1}{8}[B \,\text{.}\, B] + \tfrac{1}{2}[B, C]\right\|^2.
\end{aligned}
$$

2.2 Scaling

The Vafa-Witten equations transform nicely under scaling the metric by a constant factor. For any $\eta \in \mathbb{R}$, consider the metric $e^{2\eta} g_0$. We show that under scaling, the space of solutions remains essentially unchanged.

We define the *scale transformation* of $\eta \in \mathbb{R}$ on the parameterized configuration space $\mathcal{C} \times \mathrm{Met}$ by

$$
\eta \cdot (C, A, B, g) \mapsto (e^{-\eta} C, A, e^{\eta} B, e^{2\eta} g).
$$

Scaling transformations act as follows:

$$\begin{aligned}
\nabla_A B &\mapsto e^{\eta} \nabla_A B, \\
\nabla_A B \otimes \nabla_A B &\mapsto e^{2\eta} \nabla_A B \otimes \nabla_A B, \\
|\nabla_A B|^2 &\mapsto e^{-4\eta} |\nabla_A B|^2, \\
|\nabla_A B|^2 \, d\mathrm{vol} &\mapsto |\nabla_A B|^2 \, d\mathrm{vol}.
\end{aligned}$$

Similarly, the following are examples of scale-invariant measures:

$$|\nabla_A B|^2 \, d\mathrm{vol}, \quad |F_A|^2 \, d\mathrm{vol}, \quad |\nabla_A C|^2 \, d\mathrm{vol}, \quad |B|^4 \, d\mathrm{vol}, \quad |C|^4 \, d\mathrm{vol}, \quad \mu_{\mathrm{grad}}, \quad \mu_{\mathrm{an}}.$$

Finally, for $A = A_0 + a$, we have the scale-invariant norm

$$e^{(k+1-4/p)\eta} \left\| \nabla^k_{A_0}(C, a, B) \right\|_{L^p}. \tag{2.8}$$

Note that $\varepsilon_{\mathrm{grad}}$ is scale-invariant. Though it's not hard to see more directly, we can use scale invariance to deduce that solutions are preserved by scale transformations since

$$\begin{aligned}
&\mathrm{VW}(e^{-\eta} C, A, e^{\eta} B, e^{2\eta} g) = 0 \\
\iff &\varepsilon_{\mathrm{grad}}(e^{-\eta} C, A, e^{\eta} B, e^{2\eta} g) = 0 \\
\iff &\varepsilon_{\mathrm{grad}}(C, A, B, g) = 0 \\
\iff &\mathrm{VW}(C, A, B, g) = 0.
\end{aligned}$$

The situation is slightly worse for conformal transformations, i.e. when $\eta \in C^\infty(X)$ is not necessarily constant. The expression

$$F_A^+ + \tfrac{1}{8}[B \mathbin{\boldsymbol{.}} B] + \tfrac{1}{2}[B, C]$$

is conformally invariant, however the other Vafa-Witten equation depends on $d\eta$. We give an explicit description of how the equations depend on the metric in Theorem 5.2.1.

2.3 Relation to Vafa and Witten's identity

In order to make contact with their notation, we wish to explain the equivalence between (2.4) and Vafa and Witten's identity (2.57) from [VW94]:

$$\begin{aligned}
\frac{|s|^2 + |k|^2}{2e^2} &= \frac{1}{2e^2} \int_X d^4x \sqrt{g} \mathrm{Tr} \Bigg(\left(F^+{}_{ij} + \frac{1}{4}[B_{ik}, B_{jl}] g^{kl} + \frac{1}{2}[C, B_{ij}] \right)^2 + \\
&\qquad + \left(D^j B_{ij} + D_i C \right)^2 \Bigg) \\
&= \frac{1}{2e^2} \int_X d^4x \sqrt{g} \mathrm{Tr} \Bigg(F^+{}_{ij}{}^2 + \frac{1}{4}(D_l B_{ij})^2 + (D_i C)^2 + \frac{1}{16}[B_{ik}, B_{jk}][B_{ir}, B_{jr}] \\
&\qquad + \frac{1}{4}[C, B_{ij}]^2 + \frac{1}{4} B_{ij} \left(\frac{1}{6}(g_{ik} g_{jl} - g_{il} g_{jk}) R + W^+{}_{ijkl} \right) B_{kl} \Bigg).
\end{aligned}$$

Their notation conflicts with ours: physicists prefer Hermitian matrices to skew-Hermitian matrices. They convert skew-Hermitian matrices to Hermitian matrices by dividing a skew-Hermitian matrix by $\sqrt{-1}$. Vafa and Witten's commutator leaves implicit the necessary factor of $\sqrt{-1}$ to obtain a Hermitian operator from a commutator. Thus we perform the substitutions

$$F^+ \mapsto \sqrt{-1}F^+, \qquad B \mapsto \sqrt{-1}B, \qquad C \mapsto \sqrt{-1}C, \qquad [X,Y] \mapsto \sqrt{-1}\,[X,Y],$$

to obtain the skew-Hermitian version of Vafa-Witten's identity:

$$\begin{aligned} |s|^2 + |k|^2 &= \int_X (-\mathrm{Tr}) \left(\left({F^+}_{ij} - \tfrac{1}{4}[B_{ik}, B_{jl}]g^{kl} - \tfrac{1}{2}[C, B_{ij}] \right)^2 + \left(D^j B_{ij} + D_i C \right)^2 \right) \\ &= \int_X (-\mathrm{Tr}) \left({F^+}_{ij}{}^2 + \tfrac{1}{4}(D_l B_{ij})^2 + (D_i C)^2 + \tfrac{1}{16}[B_{ik}, B_{jk}][B_{ir}, B_{jr}] \right. \\ &\qquad \left. + \tfrac{1}{4}[C, B_{ij}]^2 + \tfrac{1}{4} B_{ij} \left(\tfrac{1}{6}(g_{ik}g_{jl} - g_{il}g_{jk})R + {W^+}_{ijkl} \right) B_{kl} \right). \end{aligned} \tag{2.9}$$

Note that $-\mathrm{Tr}(C_1 C_2)$ is positive-definite on skew-Hermitian matrices, so we can identify it (up to a positive constant) with our positive-definite inner product $\langle C_1 C_2 \rangle$.

Theorem 2.3.1. *On an oriented four-manifold without boundary, in an orthonormal frame, the following identity holds:*

$$\begin{aligned} &\tfrac{1}{2} \int_X \left\langle \left((F_A^+)_{ij} - \tfrac{1}{4}\left[B_{ik}, B_{jk}\right] - \tfrac{1}{2}\left[C, B_{ij}\right] \right)^2 + \left((\nabla_{A,j} B)_{ij} + \nabla_{A,i} C \right)^2 \right\rangle \\ &= \tfrac{1}{2} \int_X \left\langle (F_A^+)_{ij}{}^2 + \tfrac{1}{4} \left(\nabla_{A,\ell} B \right)_{ij}{}^2 + \left(\nabla_{A,i} C \right)^2 + \tfrac{1}{16}\left[B_{ik}, B_{jk}\right]\left[B_{ir}, B_{jk}\right] + \right. \\ &\quad \left. + \tfrac{1}{4}\left[C, B_{ij}\right]^2 + \tfrac{1}{4}\left(\tfrac{1}{6} s (\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}) - W^+_{ijk\ell} \right) B_{ij} B_{k\ell} \right\rangle. \end{aligned}$$

This expression should be compared with the skew-Hermitian version of Vafa-Witten's expression, (2.9). The only significant difference is the opposite sign convention for $W^+_{ijk\ell}$.

Proof. It suffices to show that (2.2) is

$$\varepsilon_{\mathrm{grad}} = \int_X \left\langle \left((F_A^+)_{ij} - \tfrac{1}{4}\left[B_{ik}, B_{jk}\right] - \tfrac{1}{2}\left[C, B_{ij}\right] \right)^2 + \left((\nabla_{A,j} B)_{ij} + \nabla_{A,i} C \right)^2 \right\rangle, \tag{2.10}$$

and (2.4) is

$$\begin{aligned} \varepsilon_{\mathrm{grad}} = \tfrac{1}{2} \int_X \Big\langle & (F_A^+)_{ij}{}^2 + \tfrac{1}{4} \left(\nabla_{A,\ell} B \right)_{ij}{}^2 + \left(\nabla_{A,i} C \right)^2 + \tfrac{1}{16} \left(\left[B_{ik}, B_{jk}\right] \right)^2 + \\ & + \tfrac{1}{4} \left(\left[C, B_{ij}\right] \right)^2 + \tfrac{1}{2} \left(\tfrac{1}{12} s (\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}) - \tfrac{1}{2} W^+_{ijk\ell} \right) B_{ij} B_{k\ell} \Big\rangle. \end{aligned} \tag{2.11}$$

Expanding out (2.10) with $B = \frac{1}{2}B_{ij}e^{ij}$ and $F_A^+ = \frac{1}{2}(F_A^+)_{ij}e^{ij}$ in an orthonormal frame,

$$\begin{aligned}
\varepsilon_{\text{grad}} &= \int_X \left\langle \tfrac{1}{2}\left(e^i\nabla_{A,i}C - \tfrac{1}{2}a_j(\nabla_{A,j}B)_{ik}e^{ik}\right)\right)^{\cdot 2} + \left(\tfrac{1}{2}(F_A^+)_{ij}e^{ij} + \tfrac{1}{32}\left[B_{ij}, B_{k\ell}\right]e^{ij}\,\raisebox{0.2ex}{\tiny$\blacksquare$}\,e^{k\ell} + \tfrac{1}{4}\left[B_{ij}, C\right]e^{ij}\right)^{\cdot 2}\right\rangle \\
&= \int_X \left\langle \tfrac{1}{2}\left(e^i\nabla_{A,i}C + e^i(\nabla_{A,j}B)_{ij}\right)^{\cdot 2} + \left(\tfrac{1}{2}(F_A^+)_{ij}e^{ij} - \tfrac{1}{8}\left[B_{ik}, B_{jk}\right]e^{ij} + \tfrac{1}{4}\left[B_{ij}, C\right]e^{ij}\right)^{\cdot 2}\right\rangle \\
&= \tfrac{1}{2}\int_X \left\langle \left((F_A^+)_{ij} - \tfrac{1}{4}\left[B_{ik}, B_{jk}\right] - \tfrac{1}{2}\left[C, B_{ij}\right]\right)^2 + \left((\nabla_{A,j}B)_{ij} + \nabla_{A,i}C\right)^2\right\rangle.
\end{aligned}$$

Similarly, for (2.11),

$$\begin{aligned}
\varepsilon_{\text{grad}} &= \int_X \left\langle \tfrac{1}{2}\left(d_A C\right)^{\cdot 2} + \left(F_A^+\right)^{\cdot 2} + \tfrac{1}{64}\left[B\,\raisebox{0.2ex}{\tiny$\blacksquare$}\,B\right]^{\cdot 2} + \tfrac{1}{4}\left(\nabla_A B\right)^{\cdot 2} + \tfrac{1}{4}\left[B, C\right]^{\cdot 2} + \tfrac{1}{4}B\cdot\left(\tfrac{1}{6}s + W^+\right)B\right\rangle \\
&= \int_X \Big\langle \tfrac{1}{2}\left(e^i\nabla_{A,i}C\right)^{\cdot 2} + \left(\tfrac{1}{2}\left(F_A^+\right)_{ij}e^{ij}\right)^{\cdot 2} + \tfrac{1}{64}\left(\tfrac{1}{2}\left[B\,\raisebox{0.2ex}{\tiny$\blacksquare$}\,B\right]_{ij}e^{ij}\right)^{\cdot 2} + \tfrac{1}{4}\left(\tfrac{1}{2}\left(\nabla_A B\right)_{ij}e^{ij}\right)^{\cdot 2} + \\
&\qquad\qquad + \tfrac{1}{4}\left(\tfrac{1}{2}\left[B_{ij}, C\right]e^{ij}\right)^{\cdot 2} + \tfrac{1}{4}B_{ij}e^{ij}\cdot\left(\tfrac{1}{12}s + \tfrac{1}{2}W^+\right)B_{k\ell}e^{k\ell}\Big\rangle \\
&= \tfrac{1}{2}\int_X \Big\langle \left(F_A^+\right)_{ij}{}^2 + \tfrac{1}{4}\left(\nabla_{A,\ell}B\right)_{ij}{}^2 + \left(\nabla_{A,i}C\right)^2 + \tfrac{1}{16}\left[B_{ik}, B_{jk}\right]\left[B_{i\ell}, B_{j\ell}\right] + \\
&\qquad\qquad + \tfrac{1}{4}\left[C, B_{ij}\right]^2 + \tfrac{1}{4}\left(\tfrac{1}{6}s(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}) - W^+_{ijk\ell}\right)B_{ij}B_{k\ell}\Big\rangle.
\end{aligned}$$

$\square$

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Chapter 3

Analytic results

3.1 An L^∞ bound on B

Using a technique of Taubes [Tau82, p. 166] (also described in [Law85, p. 76]), we combine the Weitzenböck formula with Morrey's mean-value inequality to deduce a bound on $\|B\|_{L^\infty}$ in terms of $\|B\|_{L^2}$.

3.1.1 The Weitzenböck estimate for B

Theorem 3.1.1. *Let X be a smooth closed oriented Riemannian manifold. There exists a constant C with the following property. For any principal bundle $P \to X$ and any L^2_1 solution $(0, A, B)$ to the Vafa-Witten equations,*

$$\|B\|_{L^\infty} \le C \|B\|_{L^2}.$$

Proof. By Theorem 3.3.8, we may assume that A and B are smooth. By the Weitzenböck formula (B.18), any solution $(0, A, B)$ satisfies

$$\nabla_A^* \nabla_A B = \left(-\tfrac{1}{3}s + 2W^+\right) B - \tfrac{1}{8}\left[[B \,.\, B] \,.\, B\right],$$

so pointwise,

$$\langle B \cdot \nabla_A^* \nabla_A B \rangle = \langle B \cdot \left(-\tfrac{1}{3}s + 2W^+\right) B \rangle - \tfrac{1}{8}\left|[B \,.\, B]\right|^2.$$

Since X is compact, we get a pointwise bound of the form

$$\langle B \cdot \nabla_A^* \nabla_A B \rangle \le \lambda |B|^2$$

for some constant λ depending only on Riemannian curvature of X. Theorem 3.1.2 yields the desired estimate

$$\|B\|_{L^\infty} \le C \|B\|_{L^2}.$$

□

3.1.2 Subharmonicity of B

Theorem 3.1.2. *Let X be a smooth closed Riemannian manifold. For all $\lambda > 0$ there are constants $\{C_\lambda\}$ with the following property. Let $V \to X$ be any real vector bundle equipped with a metric, $A \in \mathcal{A}_V$ any smooth metric-compatible connection, and $\sigma \in \Omega^0(X; V)$ any smooth section which satisfies the pointwise inequality*

$$\langle \sigma \cdot \nabla_A^* \nabla_A \sigma \rangle \le \lambda \left|\sigma\right|^2 .$$

Then σ satisfies the estimate

$$\left\|\sigma\right\|_{L^\infty}^2 \le C_\lambda \left\|\sigma\right\|_{L^2}^2 .$$

Proof. Naïvely, our goal is to apply Corollary 3.1.6 with $u = |\sigma|$. Two complications arise: it doesn't satisfy the hypothesis $u \ge 1$, and u is not necessarily smooth where $\sigma = 0$. Instead, in Lemma 3.1.3 we show that

$$\Delta\sqrt{1 + \left|\sigma\right|^2} \le \lambda\sqrt{1 + \left|\sigma\right|^2}.$$

Thus Corollary 3.1.6 applies to $u = \sqrt{1 + \left|\sigma\right|^2}$, and we obtain

$$\left\|\sigma\right\|_{L^\infty}^2 = \left\|\sqrt{1 + \left|\sigma\right|^2}\right\|_{L^\infty}^2 - 1 \le C_\lambda \left\|\sqrt{1 + \left|\sigma\right|^2}\right\|_{L^2}^2 - 1 = \left(C_\lambda \mathrm{vol}(X) - 1\right) + C_\lambda \left\|\sigma\right\|_{L^2}^2 .$$

We dispense with the constant term via homogeneity. For any constant $\alpha > 0$, we have the following sequencce of implications:

$$\begin{aligned}
\langle \sigma \cdot \nabla_A^* \nabla_A \sigma \rangle &\le \lambda \left|\sigma\right|^2 , \\
\langle \alpha\sigma \cdot \nabla_A^* \nabla_A \alpha\sigma \rangle &\le \lambda \left|\alpha\sigma\right|^2 , \\
\left\|\alpha\sigma\right\|_{L^\infty}^2 &\le \left(C_\lambda \mathrm{vol}(X) - 1\right) + C_\lambda \left\|\alpha\sigma\right\|_{L^2}^2 , \\
\left\|\sigma\right\|_{L^\infty}^2 &\le \left(C_\lambda \mathrm{vol}(X) - 1\right)/\alpha^2 + C_\lambda \left\|\sigma\right\|_{L^2}^2 .
\end{aligned}$$

Taking $\alpha \to \infty$, we get

$$\left\|\sigma\right\|_{L^\infty}^2 \le C_\lambda \left\|\sigma\right\|_{L^2}^2 .$$

□

Lemma 3.1.3. *Suppose that f is a function such that*

$$\begin{aligned}
f &\in C^2([0, \infty)), \\
f &\ge 0, \\
f'(x) &> 0, \\
1 + 2x\frac{f''(x)}{f'(x)} &\ge 0..
\end{aligned}$$

(For example, the hypotheses are satisfied for $f(x) = \sqrt{k^2 + x}$ with any positive constant k.) For any vector bundle $V \to X$ with metric connection A, and any section $\sigma \in \Omega^0(X; V)$, we define

$$s := f(\left|\sigma\right|^2).$$

Then

$$\tfrac{1}{2}\Delta s \le \left\langle f'(|\sigma|^2)\sigma \cdot \nabla_A^*\nabla_A\sigma \right\rangle .$$

In the case $f = \sqrt{k^2 + x}$, *we get*

$$\Delta\sqrt{k^2 + |\sigma|^2} \le \frac{\langle \sigma \cdot \nabla_A^*\nabla_A\sigma\rangle}{k^2 + |\sigma|^2}\sqrt{k^2 + |\sigma|^2}.$$

Proof. We compute

$$\begin{aligned}
\tfrac{1}{2}\nabla s &= \left\langle f'(|\sigma|^2)\sigma \cdot \nabla_A\sigma \right\rangle , \\
\tfrac{1}{2}\Delta s &= f'(|\sigma|^2)\left(\langle \sigma \cdot \nabla_A^*\nabla_A\sigma\rangle - \left(|\nabla_A\sigma|^2 + 2\,|\sigma|^2\,\frac{f''(|\sigma|^2)}{f'(|\sigma|^2)}\,|\nabla|\sigma||^2\right)\right). \\
&\le \left\langle f'(|\sigma|^2)\sigma \cdot \nabla_A^*\nabla_A\sigma \right\rangle - f'(|\sigma|^2)\,|\nabla|\sigma||^2\left(1 + 2\,|\sigma|^2\,\frac{f''(|\sigma|^2)}{f'(|\sigma|^2)}\right) \\
&\le \left\langle f'(|\sigma|^2)\sigma \cdot \nabla_A^*\nabla_A\sigma \right\rangle .
\end{aligned}$$

In the case $s = \sqrt{k^2 + |\sigma|^2}$, the inequality becomes

$$\Delta\sqrt{k^2 + |\sigma|^2} \le \left\langle \frac{\sigma}{\sqrt{k^2 + |\sigma|^2}} \cdot \nabla_A^*\nabla_A\sigma \right\rangle = \frac{\langle \sigma \cdot \nabla_A^*\nabla_A\sigma\rangle}{k^2 + |\sigma|^2}\sqrt{k^2 + |\sigma|^2}.$$

□

3.1.3 Morrey's mean-value inequality

We reproduce the full statement of [Mor66, Theorem 5.3.1, p. 137] in Theorem 3.1.5. Then we adapt the result for the Laplacian on a manifold in Corollary 3.1.6.

3.1.3.1 The weak Laplacian on a domain

Let $G \subset \mathbb{R}^\nu$ be a bounded domain with $\nu \ge 3$. Let α, β denote indices ranging from 1 to ν. Suppose we have a collection of functions

$$a^{\alpha\beta} \in L^\infty(G), \quad b^\alpha \in L^\nu(G), \quad c^\alpha \in L^\nu(G), \quad d \in L^{\nu/2}(G).$$

Suppose further that there exist positive constants m, M, C_0, μ_1 such that

$$\begin{aligned} m\,|\lambda|^2 &\le a^{\alpha\beta}(x)\lambda_\alpha\lambda_\beta \text{ for all } \lambda \in \mathbb{R}^\nu \text{ and a.e. } x \in G, \\ \|a\|_{L^\infty(G)} &\le M, \\ P &:= \sqrt{|b|^2 + |c|^2 + |d|} \in L^n(G) \\ \|P\|^2_{L^n(B(x_0,r)\cap G)} &\le C_0 r^{\mu_1} \text{ for every } x_0 \in G,\ r > 0. \end{aligned} \tag{3.1}$$

Definition 3.1.4. Given functions $a^{\alpha\beta}, b^\alpha, c^\alpha, d$ satisfying (3.1), we define an operator Δ_{abcd} based on the formula

$$\Delta_{abcd} = \text{“} - \partial_\alpha(a^{\alpha\beta}\partial_\beta + b^\alpha) + c^\alpha\partial_\alpha + d\text{”}.$$

More precisely, for $u \in L^1_{1,\mathrm{loc}}$, we define $\Delta_{abcd}(u)$ to be the distribution given by

$$\zeta \cdot \Delta_{abcd}(u) := \int_G \big((\partial_\alpha\zeta)(a^{\alpha\beta}\partial_\beta u + b^\alpha u) + \zeta(c^\alpha\partial_\alpha u + du)\big) \text{ for all } \zeta \in C_c^\infty(G).$$

For $p \in [1,\nu)$, this gives a continuous map

$$\Delta_{abcd} : L^p_{1,\mathrm{loc}}(G) \to L^p_{-1,\mathrm{loc}}(G),$$

i.e. our test function ζ makes sense when extended to the dual space

$$\big(L^p_{-1,\mathrm{loc}}(G)\big)^* = L^{p/(p-1)}_{1,c}(G) \quad \text{or} \quad \big(L^1_{-1,\mathrm{loc}}\big)^* = L^\infty_{1,c} = \mathrm{Lip}_c(G).$$

3.1.3.2 Morrey's mean-value inequality

Theorem 3.1.5 ([Mor66, Theorem 5.3.1, p. 137])**.** *For any fixed integer $\nu \ge 3$ and for any positive real numbers $m, M, C_0, \mu_1, \lambda$, there exists a constant C such that for any bounded domain $G \subset \mathbb{R}^\nu$ and for any functions $a^{\alpha\beta}, b^\alpha, c^\alpha, d, U$ on G which satisfy* (3.1) *and*

$$\begin{aligned} U &\in L^2_{1,\mathrm{loc}}(G) \cap L^2(G), \\ U &\ge 1, \\ \Delta_{abcd}(U^\lambda) &\le 0 \text{ for some } \lambda \in [1,2), \end{aligned}$$

there are estimates

$$\|U\|^2_{L^\infty(B(x_0,R))} \le Ca^{-\nu}\,\|U\|^2_{L^2(B(x_0,R+a))}$$

for all x_0, R, a such that $a \in (0,R]$ and $B(x_0, R+a) \subset G$.

3.1.3.3 The Laplacian on a manifold

Corollary 3.1.6. *Let (X,g) be a smooth Riemannian four-manifold of dimension $n \ge 3$. Suppose $\Omega \Subset X$ is a precompact domain with smooth boundary. There exist positive constants $R_{\max}$ and $\{C_\lambda\}_{\lambda\in\mathbb{R}}$*

such that for any constant $\lambda \in \mathbb{R}$ and any function $u \in L^2_{1,\mathrm{loc}}(\Omega) \cap L^2(\Omega)$ satisfying

$$\begin{aligned} u &\geq 1, \\ \Delta u &\leq \lambda u, \end{aligned}$$

there are estimates

$$\|u\|^2_{L^\infty(B(x,R))} \leq C^{\mathrm{loc}}_\lambda a^{-n} \|u\|^2_{L^2(B(x,R+a))}$$

for all $x \in U$ and all $R, a \in \mathbb{R}$ such that both $0 < a \leq R \leq R_{\max}$ and $B(x, R + a) \subset \Omega$.

Furthermore, if X is closed, then we can find C_λ such that

$$\|u\|^2_{L^\infty(X)} \leq C_\lambda \|u\|^2_{L^2(X)}.$$

Proof. Recall that in a coordinate chart,

$$\Delta = -\frac{1}{\sqrt{g}} \partial_i \sqrt{g} g^{ij} \partial_j.$$

We obtain $\Delta_{abcd} = \Delta - \lambda$ from Definition 3.1.4 if we set

$$\begin{aligned} a^{\alpha\beta} &= g^{\alpha\beta}, \\ b^\alpha &= 0, \\ c^\alpha &= -\frac{1}{\sqrt{g}} g^{\alpha\beta} (\partial_\beta \sqrt{g}), \\ d &= -\lambda. \end{aligned}$$

Thus

$$\Delta u \leq \lambda u \iff \Delta_{abcd} u \leq 0.$$

Since Ω is precompact, we can find finitely many geodesic coordinate balls $\{B(x_i, 4R_i)\}$ such that the $\{B(x_i, R_i)\}$ cover Ω, and each $B(x_i, 3R_i) \cap \Omega$ is connected. Set $R_{\max} = \min\{R_i\}$. Since the metric is smooth, we can find constants m, M, C_0, μ_1 satisfying (3.1) for all $B(x_i, 3R_i) \cap \Omega$ simultaneously. (Note that the C_0 of (3.1) depends on λ.) We take C^{loc}_λ as the corresponding constant C of Theorem 3.1.5.

Any ball $B(x, R + a)$ with $0 < a \leq R \leq R_{\max}$ is contained in some $B(x_i, 3R_i)$. Thus if $B(x, R + a) \subset \Omega$, we get the desired estimate

$$\|u\|^2_{L^\infty(B(x,R))} \leq C^{\mathrm{loc}}_\lambda a^{-n} \|u\|^2_{L^2(B(x,R+a))}$$

in the coordinate chart for $B(x_i, 3R_i)$.

In the case when X is closed, we may cover X with finitely many geodesic balls $B(y_i, R_{\max})$. Then

$$\|u\|^2_{L^\infty(X)} \leq \max\left\{\|u\|^2_{L^\infty(B(y_i,R_{\max}))}\right\} \leq \max\left\{C^{\mathrm{loc}}_\lambda R^{-n}_{\max} \|u\|^2_{L^2(B(y_i,2R_{\max}))}\right\} \leq C^{\mathrm{loc}}_\lambda R^{-n}_{\max} \|u\|^2_{L^2(X)},$$

so we may take $C_\lambda = C^{\mathrm{loc}}_\lambda R^{-n}_{\max}$. □

3.2 Coulomb slices

3.2.1 The Kuranishi complex

The most fundamental tool for understanding $\mathcal{M}_{\mathrm{ASD}}$ is the complex associated to an ASD connection A given by

$$0 \longrightarrow \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^+} \Omega^{2,+}(\mathfrak{g}_P) \longrightarrow 0. \tag{3.2}$$

The cohomology groups $H_A^\bullet$ each have a geometric interpretation: H_A^0 detects reducible connections, H_A^1 is the tangent space $T_A\mathcal{M}_{\mathrm{ASD}}$, and H_A^2 measures the failure of transversality. In this section, we study the corresponding complex for the Vafa-Witten equations.

The complex associated to the Vafa-Witten equations is of the form

$$0 \longrightarrow \Omega^0(\mathfrak{g}_P) \xrightarrow{d^0_{(C,A,B)}} \Omega^0(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P) \times \Omega^{2,+}(\mathfrak{g}_P) \xrightarrow{d^1_{(C,A,B)}} \Omega^1(\mathfrak{g}_P) \times \Omega^{2,+}(\mathfrak{g}_P) \longrightarrow 0,$$

where $d^1_{(C,A,B)}$ is the linearization of VW at the configuration (C,A,B), and $d^0_{(C,A,B)}$ gives the action of infinitesimal gauge transformations. These maps $d^0_{(C,A,B)}$ and $d^1_{(C,A,B)}$ form a complex whenever $\mathrm{VW}(C,A,B) = 0$.

The action of $g \in \mathcal{G}_P$ on $(C,A,B) \in \Omega^0(\mathfrak{g}_P) \times \mathcal{A}_P \times \Omega^{2,+}(\mathfrak{g}_P)$ is given by

$$(C,A,B) \mapsto (gCg^{-1}, A - (d_A g)g^{-1}, gBg^{-1}).$$

The Lie algebra of $\mathcal{G}_P$ is $\Omega^0(\mathfrak{g}_P)$, and the corresponding infinitesimal action of $\xi \in \Omega^0(\mathfrak{g}_P)$ is

$$\begin{aligned} d^0_{(C,A,B)} &: \Omega^0(\mathfrak{g}_P) \to \Omega^0(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P) \oplus \Omega^{2,+}(\mathfrak{g}_P), \\ d^0_{(C,A,B)}(\xi) &= \begin{pmatrix} [\xi, C] \\ -d_A\xi \\ [\xi, B] \end{pmatrix}. \end{aligned}$$

The linearization of VW at the point (C,A,B) is given by

$$d^1_{(C,A,B)} \begin{pmatrix} c \\ a \\ b \end{pmatrix} = \begin{pmatrix} d_A c & -[C,a] - [B \,\raisebox{0.2ex}{$\centerdot$}\, a] & +d_A^* b \\ \frac{1}{2}[B,c] & +d_A^+ a & +\frac{1}{4}[B \,\raisebox{0.2ex}{$\centerdot$}\, b] - \frac{1}{2}[C,b] \end{pmatrix}.$$

The reason $d^0_{(C,A,B)}$ and $d^1_{(C,A,B)}$ form a complex is roughly that gauge transformations preserve solutions. The following equivariance condition on VW is a consequence of the "gauge principle" that each individual operator in the expression for VW is gauge-equivariant:

$$\mathrm{VW}(gCg^{-1}, A - (d_A g)g^{-1}, gBg^{-1}) = \begin{pmatrix} g(d_A C + d_A^* B)g^{-1} \\ g(F_A^+ + \frac{1}{8}[B \,\raisebox{0.2ex}{$\centerdot$}\, B] + \frac{1}{2}[B,C])g^{-1} \end{pmatrix}. \tag{3.3}$$

By differentiating (3.3) we obtain

$$d^1_{(C,A,B)} \circ d^0_{(C,A,B)}(\xi) = \begin{pmatrix} [\xi, d_A C + d_A^* B] \\ [\xi, F_A^+ + \frac{1}{8}[B \centerdot B] + \frac{1}{2}[B, C]] \end{pmatrix}.$$

Alternatively, it's simple to compute this composition directly with assistance of the identities

$$\begin{aligned} d_A^*[\xi, B] &= -[d_A \xi \centerdot B] + [\xi, d_A^* B], \\ [\xi, [B \centerdot B]] &= 2[[\xi, B] \centerdot B]. \end{aligned}$$

We compute

$$\begin{aligned} d^1_{(C,A,B)} \circ d^0_{(C,A,B)}(\xi) &= \begin{pmatrix} d_A[\xi, C] + d_A^*[\xi, B] - [d_A \xi, C] + [d_A \xi \centerdot B] \\ -d_A^+ d_A \xi + \frac{1}{4}[[\xi, B] \centerdot B] + \frac{1}{2}[[\xi, B], C] - \frac{1}{2}[[\xi, C], B] \end{pmatrix} \\ &= \begin{pmatrix} [\xi, d_A C + d_A^* B] \\ [\xi, F_A^+ + \frac{1}{8}[B \centerdot B] + \frac{1}{2}[B, C]] \end{pmatrix}. \end{aligned}$$

The dual complex is

$$0 \longrightarrow \Omega^1(\mathfrak{g}_P) \times \Omega^{2,+}(\mathfrak{g}_P) \xrightarrow{d^{1,*}_{(C,A,B)}} \Omega^0(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P) \times \Omega^{2,+}(\mathfrak{g}_P) \xrightarrow{d^{0,*}_{(C,A,B)}} \Omega^0(\mathfrak{g}_P) \longrightarrow 0.$$

These codifferentials are

$$d^{1,*}_{(C,A,B)} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} d_A^* \hat{a} & +\frac{1}{2}[\hat{b} \cdot B] \\ -[\hat{a}, C] - [\hat{a} \centerdot B] & +d_A^* \hat{b} \\ d_A^+ \hat{a} & +\frac{1}{4}[\hat{b} \centerdot B] - \frac{1}{2}[\hat{b}, C] \end{pmatrix},$$

and

$$d^{0,*}_{(C,A,B)} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\begin{pmatrix} [c, C] & +d_A^* a & +[b \cdot B] \end{pmatrix}.$$

Again we verify the composite

$$\begin{aligned} d^{0,*}_{(C,A,B)} \circ d^{1,*}_{(C,A,B)} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} &= -\left[d_A^* \hat{a} + \tfrac{1}{2}[\hat{b} \cdot B], C\right] - d_A^* \left(d_A^* \hat{b} - [\hat{a}, C] - [\hat{a} \centerdot B]\right) + \\ &\quad - \left[\left(d_A^+ \hat{a} + \tfrac{1}{4}[\hat{b} \centerdot B] - \tfrac{1}{2}[\hat{b}, C]\right) \cdot B\right] \\ &= -\left([\hat{a} \cdot (d_A C + d_A^* B)] + \left[\hat{b} \cdot (F_A^+ + \tfrac{1}{8}[B \centerdot B] + \tfrac{1}{2}[B, C])\right]\right), \end{aligned}$$

by using the identities

$$\begin{aligned} d_A^* d_A^* \hat{b} &= -[F_A^+ \cdot \hat{b}], \\ [\hat{b} \cdot [B \centerdot B]] &= 2[[\hat{b} \centerdot B] \cdot B], \\ d_A^*[\hat{a}, C] &= [d_A^* \hat{a}, C] - [\hat{a} \cdot d_A C], \\ d_A^*[\hat{a} \centerdot B] &= [d_A^+ \hat{a} \cdot B] - [\hat{a} \cdot d_A^* B]. \end{aligned}$$

3.2.2 The quadratic expansion

Theorem 3.2.1. *The map* $\mathrm{VW}(C,A,B)$ *has an exact quadratic expansion given by*

$$\mathrm{VW}(C+c,A+a,B+b)=\mathrm{VW}(C,A,B)+d^1_{(C,A,B)}(c,a,b)+\{(c,a,b),(c,a,b)\},$$

where $\{(c,a,b),(c,a,b)\}$ *is the symmetric quadratic form given by*

$$\{(c,a,b),(c,a,b)\}:=[a,c]-[a\,\boldsymbol{.}\,b]\oplus\tfrac{1}{2}[a\wedge a]^{+}+\tfrac{1}{8}[b\,\boldsymbol{.}\,b]+\tfrac{1}{2}[b,c].$$

Proof. Expanding,

$$\begin{aligned}\mathrm{VW}(C+c,A+a,B+b)&=d_{A+a}(C+c)+d^*_{A+a}(B+b)\\&\quad\oplus F^+_{A+a}+\tfrac{1}{8}[(B+b)\,\boldsymbol{.}\,(B+b)]+\tfrac{1}{2}[(B+b),(C+c)]\\&=\mathrm{VW}(C,A,B)+\\&\quad+[a,c]-[a\,\boldsymbol{.}\,b]+[a,C]+d_Ac-[a\,\boldsymbol{.}\,B]+d^*_Ab\\&\quad\oplus d^+_Aa+\tfrac{1}{2}[a\wedge a]^{+}+\tfrac{1}{8}[b\,\boldsymbol{.}\,b]+\tfrac{1}{2}[b,c]+\tfrac{1}{4}[B\,\boldsymbol{.}\,b]+\tfrac{1}{2}[B,c]+\tfrac{1}{2}[b,C]\\&=\mathrm{VW}(C,A,B)+d^1_{(C,A,B)}(c,a,b)+\\&\quad+[a,c]-[a\,\boldsymbol{.}\,b]\oplus\tfrac{1}{2}[a\wedge a]^{+}+\tfrac{1}{8}[b\,\boldsymbol{.}\,b]+\tfrac{1}{2}[b,c].\end{aligned}$$

$\square$

3.2.3 The slice theorem

Given fixed (C_0,A_0,B_0), we look for solutions to the inhomogeneous equation $\mathrm{VW}(C_0+c,A_0+a,B_0+b)=\psi_0$. By Theorem 3.2.1, this equation is equivalent to

$$d^1_{(C_0,A_0,B_0)}(c,a,b)+\{(c,a,b),(c,a,b)\}=\psi_0-\mathrm{VW}(C_0,A_0,B_0). \tag{3.4}$$

To make this equation elliptic, it's natural to impose the (inhomogeneous) gauge-fixing condition

$$d^{0,*}_{(C_0,A_0,B_0)}(c,a,b)=\zeta.$$

If we define

$$\begin{aligned}\mathcal{D}_{(C_0,A_0,B_0)}&:=d^{0,*}_{(C_0,A_0,B_0)}+d^1_{(C_0,A_0,B_0)},\\\psi&:=\psi_0-\mathrm{VW}(C_0,A_0,B_0),\end{aligned}$$

then the elliptic system can be rewritten as

$$\mathcal{D}_{(C_0,A_0,B_0)}(c,a,b)+\{(c,a,b),(c,a,b)\}=(\zeta,\psi). \tag{3.5}$$

This is situation is considered in [FL98, eq. (3.2)] in the context of $\mathrm{PU}(2)$ monopoles. Many of their theorems apply in general to any gauge-invariant equation of the form (3.4) such that $d^{0,*}+d^1$ is elliptic.

First we define a slice for the quotient space $\mathcal{B}_P$ completed to L^2_k with $k \geq 2$.

Definition 3.2.2. For $k \geq 2$, and $(C, A, B) \in \mathcal{C}_P^{L^2_k}$, we define the ball

$$\mathbf{B}^{L^2_k}_{C,A,B}(\epsilon) := (C, A, B) + \left\{ (c, a, b) \mid d^{0,*}_{(C,A,B)}(c, a, b) = 0, \ \|(c, a, b)\|_{L^2_{k,A}} < \epsilon \right\}.$$

Under the restriction $k \geq 2$, we get a slice theorem over a closed manifold:

Theorem 3.2.3 ([FL98, Proposition 2.8])**.** *Let X be a closed, oriented, Riemannian four-manifold, let $P \longrightarrow X$ be a principal bundle with compact structure group G, and let $k \geq 2$ be an integer. Then the following hold.*

1. The space $\mathcal{B}_P^{L^2_k}$ is Hausdorff;
2. The subspace $\mathcal{B}_P^{*,L^2_k} \subset \mathcal{B}_P^{L^2_k}$ of (C_0, A_0, B_0) such that $\mathrm{Stab}_{C_0,A_0,B_0} = \mathrm{Center}(G)$ is open and is a C^∞ Hilbert manifold with local parameterizations given by $\pi : \mathbf{B}^{L^2_k}_{C_0,A_0,B_0}(\epsilon) \to \mathcal{B}_P^{L^2_k}$ for sufficiently small $\epsilon = \epsilon(C_0, A_0, B_0, k)$;
3. The projection $\pi : \mathcal{C}_P^{*,L^2_k} \to \mathcal{B}_P^{*,L^2_k}$ is a principal $\mathcal{G}_P$ bundle;
4. For $(C_0, A_0, B_0) \in \mathcal{C}$, the projection $\pi : \mathbf{B}_{C_0,A_0,B_0}(\epsilon)/\mathrm{Stab}_{C_0,A_0,B_0} \to \mathcal{B}$ is a homeomorphism onto an open neighborhood of $[C_0, A_0, B_0] \in \mathcal{B}$ and a diffeomorphism on the complement of the set of points in $\mathbf{B}_{C_0,A_0,B_0}(\epsilon)$ with non-trivial stabilizer.

Alternatively, we can restate this as the charts

$$(\mathbf{B}^{L^2_k}_{C_0,A_0,B_0}(\epsilon) \times \mathcal{G}^{L^2_{k+1}})/\mathrm{Stab}_{C_0,A_0,B_0} \to \mathcal{C}^{L^2_k}$$

being $\mathcal{G}^{L^2_k}$-equivariant diffeomorphisms onto their images, and covering $\mathcal{C}^{L^2_k}$.

3.3 Regularity and elliptic estimates

First we summarize the results of [FL98, §3], which apply almost verbatim to the Vafa-Witten equations upon replacing the PU(2) spinor Φ by the pair (C, B).

3.3.1 Global estimate for L^2_1 solutions to the inhomogeneous Vafa-Witten plus Coulomb slice equations

Theorem 3.3.1 ([FL98, Corollary 3.4])**.** *Let X be a closed, oriented, Riemannian four-manifold, let $P \longrightarrow X$ be a principal bundle with compact structure group, and let (C_0, A_0, B_0) be a C^∞ configuration in $\mathcal{C}_P$. Then there is a positive constant $\epsilon = \epsilon(C_0, A_0, B_0)$ such that if (c, a, b) is an L^2_1 solution to* (3.5) *over X, where (ζ, ψ) is in L^2_k and $\|(c, a, b)\|_{L^4} < \epsilon$, and $k \geq 0$ is an integer, then $(c, a, b) \in L^2_{k+1}$ and* *there is a polynomial $Q_k(x,y)$, with positive real coefficients, depending at most on (C_0, A_0, B_0), k, such that $Q_k(0,0) = 0$ and*

$$\|(c,a,b)\|_{L^2_{k+1,A_0}(X)} \leq Q_k\left(\|(\zeta,\psi)\|_{L^2_{k,A_0}(X)}, \|(c,a,b)\|_{L^2(X)}\right).$$

In particular, if (ζ,ψ) is in C^∞ then (c,a,b) is in C^∞ and if $(\zeta,\psi) = 0$, then

$$\|(c,a,b)\|_{L^2_{k+1,A_0}(X)} \leq C\,\|(c,a,b)\|_{L^2(X)}.$$

3.3.2 Global regularity of L^2_k solutions to the Vafa-Witten equations for $k \geq 2$

Theorem 3.3.2 ([FL98, Proposition 3.7]). *Let X be a closed, oriented, Riemannian four-manifold, and let $P \to X$ be a principal bundle with compact structure group. Let $k \geq 2$ be an integer and suppose that (C,A,B) is an L^2_k solution to $\mathrm{VW}(C,A,B) = 0$. Then there is a $\mathrm{g} \in \mathcal{G}_P^{L^2_{k+1}}$ such that $\mathrm{g}(C,A,B)$ is C^∞ over X.*

3.3.3 Local interior estimate for L^2_1 solutions to the inhomogeneous Vafa-Witten plus Coulomb slice equations

Theorem 3.3.3 ([FL98, Corollary 3.11]). *Let X be a closed, oriented, Riemannian four-manifold, and let $P \to X$ be a principal bundle with compact structure group. Suppose $\Omega \subset X$ is an open subset such that $P|_\Omega$ is trivial, and Γ is a smooth flat connection. Then there is a positive constant $\epsilon = \epsilon(\Omega)$ with the following significance. Suppose that (c,a,b) is an $L^2_1(\Omega)$ solution to the elliptic system* (3.5) *over Ω with $(C_0,A_0,B_0) = (0,\Gamma,0)$, where (ζ,ψ) is in $L^2_k(\Omega)$, $k \geq 1$ is an integer, and $\|(c,a,b)\|_{L^4(\Omega)} < \epsilon$. Let $\Omega' \Subset \Omega$ be a precompact open subset. Then (a,ϕ) is in $L^2_{k+1}(\Omega')$ and there is a universal polynomial $Q_k(x,y)$, with positive real coefficients, depending at most on k, Ω', Ω, such that $Q_k(0,0) = 0$ and*

$$\|(c,a,b)\|_{L^2_{k+1,\Gamma}(\Omega')} \leq Q_k\left(\|(\zeta,\psi)\|_{L^2_{k,\Gamma}(X)}, \|(c,a,b)\|_{L^2(X)}\right).$$

If (ζ,ψ) is in $C^\infty(\Omega)$ then (c,a,b) is in $C^\infty(\Omega')$ and if $(\zeta,\psi) = 0$, then

$$\|(c,a,b)\|_{L^2_{k+1,\Gamma}(\Omega')} \leq C\,\|(c,a,b)\|_{L^2(\Omega)}.$$

3.3.4 Local estimates for L^2_1 solutions to the Vafa-Witten equations on a ball

Theorem 3.3.4 (Uhlenbeck's gauge-fixing, [Weh04, §6], [Uhl82, Theorem 2.1 & Corollary 2.2], [FL98, Theorem 3.13]). *Let (X, g_0) be a Riemannian four-manifold without boundary, let $P \to X$ be a principal bundle with compact structure group, and let $2 \leq p < 4$. Let $D_{x,r}$ denote the geodesic ball of radius r centered at x. Then there exists constants $C, \epsilon > 0$ such that the following holds:*

For every point $x \in X$, there exists a positive radius r_x such that for all $r \in (0, r_x]$, all smooth flat connections $\Gamma \in \mathcal{A}_P(D_{x,r})$, and all L^p_1 connections $A \in \mathcal{A}_P^{L^p_1}(D_{x,r})$ with $\|F_A\|_{L^p(D_{x,r})} \leq \epsilon$, there exists a *gauge transformation* $g \in \mathcal{G}_P^{L_2^2}(D_{x,r})$ *such that*

$$\begin{aligned} d_\Gamma^*(gA - \Gamma) &= 0, \\ (gA - \Gamma)^\perp &= 0 \text{ on } \partial D_{x,r}, \\ \|gA - \Gamma\|_{L_1^p(D_{x,r})} &\le C \|F_A\|_{L^p(D_{x,r})}. \end{aligned}$$

Furthermore, if A *is in* $L_k^p(D_{x,r})$ *for* $k \ge 2$, *then* g *is in* $L_{k+1}^p(D_{x,r})$. *The gauge transformation* g *is unique up to multiplication by a constant element of* G.

At this point, we must deviate slightly from [FL98], since we have no estimate of the form $|B|^4 \le C\,|[B \,.\, B]|^2$ (c.f. [FL98, Lemma 2.26]), so F_A^+ does not bound B. Instead, we get the following analogue of [FL98, Corollary 3.15] by combining Theorem 3.3.4 and Theorem 3.3.3.

Theorem 3.3.5. *Let* $D_{0,1} \subset \mathbb{R}^4$ *be the open unit ball with center at the origin, let* $U \Subset D_{0,1}$ *be an open subset, let* $P \to D_{0,1}$ *be a principal bundle with compact structure group, and let* Γ *be a smooth flat connection on* P. *Then there is a positive constant* ϵ *such that for all integers* $k \ge 1$ *there is a constant* $C(k, U)$ *such that for all* L_1^2 *solutions* (C, A, B) *satisfying*

$$\|F_A\|_{L^2(D_{0,1})}^2 + \|B\|_{L^4(D_{0,1})}^4 + \|C\|_{L^4(D_{0,1})}^4 < \epsilon,$$

there is an L_2^2 *gauge transformation* g *such that* $g(C, A, B)$ *is in* $C^\infty(D_{0,1})$ *with* $d^*(gA - \Gamma) = 0$ *over* $D_{0,1}$ *and*

$$\|g(C, A, B)\|_{L_{k,\Gamma}^2(U)} \le C\left(\|C\|_{L^2(D_{0,1})} + \|F_A\|_{L^2(D_{0,1})} + \|B\|_{L^2(D_{0,1})}\right).$$

Proof. By choosing ϵ as in Theorem 3.3.4, we can find g such that $d_\Gamma^*(gA - \Gamma) = 0$ and $\|gA - \Gamma\|_{L_1^2(D_{0,1})} \le C\|F_A\|_{L^2(D_{0,1})}$. By the Sobolev embedding theorem, $\|gA - \Gamma\|_{L^4(D_{0,1})} \le C\|F_A\|_{L^2(D_{0,1})}$. Upon taking $(c, a, b) = (C, gA - \Gamma, B)$, we are in the situation of Theorem 3.3.3. Thus we get the desired estimate. □

Upon adding the proper factors to make this estimate scale-invariant (see (2.8)), we generalize this estimate for geodesic balls:

Theorem 3.3.6. *Let* X *be an oriented Riemannian four-manifold without boundary, and let* $P \to X$ *be a principal bundle with compact structure group. Let* $D_{x,r}$ *denote the geodesic ball of radius* r *centered at* x, *and fix any* $\alpha \in (0, 1)$. *For all* $k \ge 1$ *there exists constants* $C(\alpha, k), \epsilon > 0$ *such that the following holds:*

For every point $x \in X$, *there exists a positive radius* r_x *such that for all* $r \in (0, r_x]$, *all smooth flat connections* $\Gamma \in \mathcal{A}_P(D_{x,r})$, *and all* L_1^2 *solutions* (C, A, B) *with*

$$\|F_A\|_{L^2(D_{x,r})}^2 + \|B\|_{L^4(D_{x,r})}^4 + \|C\|_{L^4(D_{x,r})}^4 < \epsilon,$$

there exists a gauge transformation $g \in \mathcal{G}_P^{L_2^2}(D_{x,r})$ *such that* $g(C, A, B)$ *is in* $C^\infty(D_{x,r})$ *with* $d^*(gA - \Gamma) = 0$ *over* $D_{x,r}$ *and*

$$r^{k-1}\|g(C, A, B)\|_{L_{k,\Gamma}^2(D_{x,\alpha r})} \le C\left(r^{-1}\|C\|_{L^2(D_{x,r})} + \|F_A\|_{L^2(D_{x,r})} + r^{-1}\|B\|_{L^2(D_{x,r})}\right).$$

3.3.5 Local estimates for L^2_1 solutions to the Vafa-Witten equations on a more general domain

Using a patching argument, we get an estimate over strongly simply connected domains, which is an analogue of [FL98, Proposition 3.18]:

Theorem 3.3.7. *Let Ω be an oriented Riemannian four-manifold without boundary, let $P \to X$ be a principal bundle with compact structure group. Then there is a positive constant $\epsilon(\Omega)$ with the following significance. For $\Omega' \Subset \Omega$ a precompact open subset and an integer $\ell \geq 1$, there is a constant $C(\ell, \Omega', \Omega)$ such that the following holds. Suppose (C, A, B) is a smooth solution over Ω such that*

$$\|F_A\|^2_{L^2(\Omega)} + \|B\|^4_{L^4(\Omega)} + \|C\|^4_{L^4(\Omega)} < \epsilon.$$

Then there is a flat connection Γ on $P|_{\Omega'}$ and a gauge transformation g over Ω' such that

$$\|\mathsf{g}(C, A - \Gamma, B)\|_{L^2_{\ell,\Gamma}(\Omega')} \leq C\left(\|C\|_{L^2(\Omega)} + \|F_A\|_{L^2(\Omega)} + \|B\|_{L^2(\Omega)}\right).$$

3.3.6 Regularity of L^2_1 solutions *not* in Coulomb gauge

We use a recent result of Isobe to show that all L^2_1 solutions to the Vafa-Witten equations are L^2_2-gauge-equivalent to a smooth solution.

Theorem 3.3.8. *Suppose X is a closed smooth Riemannian four-manifold, $P \to X$ is a smooth principal G-bundle with G compact and connected, (C, A, B) is an L^2_1 configuration (not necessarily in Coulomb gauge!), and $\mathrm{VW}(C, A, B) = 0$. Then (C, A, B) is L^2_2-gauge-equivalent to a smooth configuration.*

Proof. By gauge-fixing on small balls D_x in which the local regularity theorem applies, we get L^2_2-trivializations $\mathsf{h}_{1,x}$ of P over D_x such that $\mathsf{h}_{1,x}(C, A, B)$ is smooth. Since the transition functions $\mathsf{h}_{1,x'}\mathsf{h}^{-1}_{1,x}$ intertwine smooth connections, they define a smooth principal G-bundle P'. The trivializations $\mathsf{h}_{1,x}$ patch together to define an L^2_2 isomorphism $\mathsf{h}_1 : P \to P'$. The $\mathsf{h}_{1,x}(C, A, B)$ determine a smooth configuration (C', A', B') in P' such that $\mathsf{h}(C, A, B) = (C', A', B')$.

In order to prove that (C, A, B) is L^2_2-gauge-equivalent to a smooth connection, it suffices to show that there exists a *smooth* isomorphism $\mathsf{h}_2 : P \to P'$, for then $\mathsf{g} := \mathsf{h}_2^{-1}\mathsf{h}_1 \in \mathcal{G}_P^{L^2_2}$ is the desired gauge transformation. The existence of h_2 is a consequence of Theorem 3.3.10. □

Towards proving Theorem 3.3.10, first we recall that the smooth classification of principal bundles is equivalent to the topological classification:

Theorem 3.3.9 ([MW06, Theorem I.13])**.** *Let X be a smooth manifold, and let G be a compact Lie group. The inclusion of sheaves $C^\infty(G) \hookrightarrow C^0(G)$ induces a bijection $\check{H}^1(X; C^\infty(G)) \to \check{H}^1(X; C^0(G))$.*

The proof is based on smooth approximation of classifying maps $[X, BG]$.

From here, there are two routes to Theorem 3.3.10. The first is specific to four dimensions, based on Sedlacek's results about characteristic classes under weak limits. The second is a recent approximation theorem for Sobolev bundles due to Isobe.

Theorem 3.3.10. *Let X be a closed smooth four-manifold equipped with two smooth principal bundles P and P' which are L^2_2-isomorphic. Then there exists a smooth isomorphism $\mathsf{h}_2 : P \to P'$.*

Proof #1. We use the classification results of [Sed82]. If G is connected, then principal bundles over X are classified by $\eta(P) \in H^2(X;\pi_1(G))$ and $p_1(P) \in H^4(X;\pi_3(G))$. By [Sed82, Theorem 5.5], $\eta(P) = \eta(P')$. By Chern-Weil theory, $p_1(P) = p_1(P')$. Thus P and P' are topologically isomorphic. By Theorem 3.3.9, they are smoothly isomorphic. □

Proof #2. The theorem is a direct corollary of the following theorem of Isobe.

Theorem 3.3.11 ([Iso09, Proposition 3.2]). *Let X be a closed smooth manifold, and let G be a compact Lie group. For any integer $k > 0$, the inclusion of sheaves $C^0(G) \hookrightarrow L^{n/k}_k(G)$, induces a bijection $\check{H}^1(X; C^0(G)) \to \check{H}^1(X; L^{n/k}_k(G))$.*

The proof of Isobe's theorem is based on the approximation theory of Sobolev maps between manifolds. □

3.4 Removal of singularities

Theorem 3.4.1 ([FL98, Theorem 4.10]). *Let $D_{x,r} \subset X$ be a geodesic ball, and let $P \to D_{x,r} \setminus \{x\}$ be a principal bundle with compact structure group. Suppose (C, A, B) is a C^∞ solution to the Vafa-Witten equations for P over the punctured ball $D_{x,r} \setminus \{x\}$ with*

$$\int_{D_{x,r}\setminus\{x\}} \left(|F_A|^2 + |\nabla_A B|^2 + |\nabla_A C|^2 + |B|^4 + |C|^4\right) d\mathrm{vol} < \infty.$$

Then there is a principal bundle $\tilde{P} \to D_{x,r}$, a C^∞ solution $(\tilde{C}, \tilde{A}, \tilde{B})$ to the Vafa-Witten equations for $\tilde{P}$ over $D_{x,r}$, and a C^∞ bundle isomorphism $u : P \to \tilde{P}|_{D_{x,r}\setminus\{x\}}$ such that

$$u^*(\tilde{C}, \tilde{A}, \tilde{B}) = (C, A, B) \text{ over } D_{x,r} \setminus \{x\}.$$

3.5 Uhlenbeck closure

Let X be a closed oriented smooth Riemannian four-manifold, and let $\{P_k \to X\}_{k\in\mathbb{Z}}$ be a collection of $\mathrm{Sp}(1)$ bundles indexed by the instanton number k. Following [FL98, §4.5.1], we define the set of ideal solutions $\mathcal{IM}_{\mathrm{VW},k}$ to be

$$\mathcal{IM}_{\mathrm{VW},k} := \bigcup_{\ell=0}^{\infty} \mathcal{M}_{\mathrm{VW},k-\ell} \times \mathrm{Sym}^\ell(X).$$

The index ℓ which appears in the above definition is called the *level*.

We say that a sequence $[C_i, A_i, B_i, \mathbf{x}_i] \in \mathcal{IM}_{\mathrm{VW},k}$ *converges to* $[C_0, A_0, B_0, \mathbf{x}_0]$ if for some (or equivalently any) choice of smooth representatives $(C_i, A_i, B_i) \in \mathcal{M}_{\mathrm{VW},k-\ell_i}$ the following hold:

- There is a sequence of smooth bundle isomorphisms $\mathrm{g}_i : P_{k-\ell_i}|_{X\backslash \mathbf{x}_0} \to P_{k-\ell_0}|_{X\backslash \mathbf{x}_0}$ such that $\mathrm{g}_i(C_i, A_i, B_i)$ converges in C^∞ to (C_0, A_0, B_0) over $X\backslash \mathbf{x}_0$.
- The sequence $|F_{A_i}|^2\, d\mathrm{vol} + 4\pi^2 \sum_{x\in\mathbf{x}_i} \delta_x$ converges in the weak-* topology on measures to $|F_{A_0}|^2 + 4\pi^2 \sum_{x\in\mathbf{x}_0} \delta_x$.

For any real constant $b \in \mathbb{R}$, we define the *b-truncated moduli space*

$$\mathcal{M}^b_{\mathrm{VW},k} := \{[0, A, B] \in \mathcal{M}_{\mathrm{VW},k} \mid \|B\|_{L^2} \leq b\}.$$

Lemma 3.5.1. The truncated moduli spaces $\mathcal{M}^b_{\mathrm{VW},k}$ satisfy

1. *$\mathcal{M}^b_{\mathrm{VW},k} \subset \mathcal{M}^{b'}_{\mathrm{VW},k}$ for $b \leq b'$.*
2. *$\mathcal{M}^0_{\mathrm{VW},k} = \mathcal{M}_{\mathrm{ASD},k}$.*
3. *$\mathcal{M}^b_{\mathrm{VW},k} = \varnothing$ for $b < 0$ or $k < -Cb^4$ for some constant C.*

Proof. All these statements are immediately obvious except for the necessity of $k \geq -Cb^4$ for $\mathcal{M}^b_{\mathrm{VW},k} \neq \varnothing$. This follows from the Chern-Weil identity, the Vafa-Witten equation (2.3), and Theorem 3.1.1:

$$0 \leq \|F^-_A\|^2 = 4\pi k + \|F^+_A\|^2 = 4\pi k + \|F^+_A\|^2 = 4\pi k + \left\|\tfrac{1}{8}[B \,\mathbf{.}\, B]\right\|^2 \leq 4\pi k + 4\pi C b^4.$$

□

Theorem 3.5.2 ([FL98, Theorem 4.20]). *Let X be a closed oriented smooth Riemannian four-manifold. Then for each $k \in \mathbb{Z}$ and $b \in \mathbb{R}$, the Uhlenbeck closure $\overline{\mathcal{M}}^b_{\mathrm{VW},k} \subset \mathcal{IM}_{\mathrm{VW},k}$ is sequentially compact.*

Proof. Since $\|B\|_{L^2}$ is bounded on $\mathcal{M}^b_{\mathrm{VW},k}$, Theorem 3.1.1 bounds $\|B\|_{L^\infty}$ and hence $\|F^+_A\|_{L^\infty}$. The level is finite by Lemma 3.5.1 . These observations are sufficient to carry through the proof of [FL98]. □

Chapter 4

Estimates for $\mathrm{SU}(2)$

4.1 Algebraic estimates

The starting point for our estimates are the identities of Section 2.1, in particular (4.1). We are interested primarily in solutions of the Vafa-Witten equations involving irreducible $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$-connections over a closed manifold (and restrictions to submanifolds thereof). Thanks to Remark 2.1.2, we need only consider solutions $(0, A, B)$ where the C component vanishes. The main difficulty in finding good estimates is that $[B \,.\, B]$ can vanish when B is nonzero.

For the Seiberg-Witten equations, the curvature bound analogous to (2.6) contains algebraic terms of the form

$$\|F_{A^t}\|^2 \leq \cdots - \int_X \left(|\Phi|^4 + s\,|\Phi|^2\right).$$

(See [KM07] §4.5) The trick to controlling this expression when the scalar curvature s is negative is to complete the square:

$$-\int_X \left(|\Phi|^4 + s\,|\Phi|^2\right) = -\int_X \left(|\Phi|^2 + \tfrac{1}{2}s\right)^2 + \tfrac{1}{4}\int_X s^2.$$

This quantity is manifestly bounded above by the geometry of X, independent of Φ. Although we cannot complete the square for the Vafa-Witten equations, by studying the algebraic properties of matrices, we show that we only lose control of $\|F_A\|$ when the L^2 density of B must accumulates in a region where B is almost rank one.

4.1.1 Matrix representations of $\Lambda^{2,+} \otimes \mathfrak{sp}(1)$

Recall the definition of the product "$\,.\,$" from A.1.6 and A.1.7. The function $B \mapsto |[B \,.\, B]|^2$ is a quartic on $\mathfrak{sp}(1) \otimes \Lambda^{2,+} T_x^* X$ which is invariant under both the adjoint action of $\mathrm{Sp}(1)$ on $\mathfrak{sp}(1)$ and the action of $\mathrm{SO}(T_x X)$ on $\Lambda^{2,+} T_x^* X$. We gain insight by studying such invariant functions.

By choosing bases $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ of $\mathfrak{sp}(1)$ and

$$\left(\sigma^1, \sigma^2, \sigma^3\right) := \left(e^{01} + e^{23}, e^{02} + e^{31}, e^{03} + e^{12}\right)$$

of $\Lambda^{2,+}$, we can represent B by a matrix

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}.$$

The action of $\mathrm{SO}(T_xX) \times \mathrm{Sp}(1)$ on $\mathfrak{sp}(1) \otimes \Lambda^{2,+}$ induces an action of $\mathrm{SO}(3) \times \mathrm{SO}(3)$ on the space of matrices $M_{3\times 3}$ by multiplication on each side separately. Singular value decomposition represents each orbit by a matrix of the form

$$B = \begin{pmatrix} B_1 & & \\ & B_2 & \\ & & B_3 \end{pmatrix}.$$

Such a matrix is unique up to permutation and flipping pairs of signs. We make the $\{B_i\}$ unique by demanding that $B_1 \geq B_2 \geq B_3 \geq 0$ or $-B_1 \geq -B_2 \geq -B_3 \geq 0$.

The singular value decomposition greatly simplifies many computations. For example,

$$\begin{aligned}
[B \,.\, B] &= \left[\left(B_1\mathbf{i}\sigma^1 + B_2\mathbf{j}\sigma^2 + B_3\mathbf{k}\sigma^3\right) . \left(B_1\mathbf{i}\sigma^1 + B_2\mathbf{j}\sigma^2 + B_3\mathbf{k}\sigma^3\right)\right] \\
&= 2\left[B_1\mathbf{i}(e^{01} + e^{23}) \,.\, B_2\mathbf{j}(e^{02} + e^{31})\right] + \text{cyclic permutations} \\
&= 2B_1B_2\,[\mathbf{i}, \mathbf{j}]\,(e^{01} + e^{23}) \,.\, (e^{02} + e^{31}) + \text{c.p.} \\
&= 2B_1B_2(2\mathbf{k})(-2)(e^{03} + e^{12}) + \text{c.p.} \\
&= \begin{pmatrix} -8B_2B_3 & & \\ & -8B_3B_1 & \\ & & -8B_1B_2 \end{pmatrix}.
\end{aligned}$$

The space of 3×3 matrices is stratified by eight families of orbits. as shown in Table 4.2 on page 39. We provide a more graphical representation in Figure 4-1 on page 41, which will be explained shortly. Note that the rank of B determines the rank of $[B \,.\, B]$, and hence F_A^+ for solutions, according to Table 4.1.

rank(B)	rank(F_A^+)
0	0
1	0
2	1
3	3

Table 4.1: Possible ranks of F_A^+, given that $0 = F_A^+ + \frac{1}{8}[B \,.\, B]$.

4.1.2 Invariant functions on $\Lambda^{2,+} \otimes \mathfrak{sp}(1)$

An invariant function on $M_{3\times 3}$ is thus equivalent to a symmetric function in B_1, B_2, B_3 which is invariant under $(B_1, B_2, B_3) \mapsto (-B_1, -B_2, B_3)$. For example, $B_1 + B_2 + B_3$ is not invariant, while

$$\alpha := B_1^2 + B_2^2 + B_3^2, \qquad \beta := (B_1B_2)^2 + (B_2B_3)^2 + (B_3B_1)^2, \qquad \gamma := B_1B_2B_3.$$

$$\begin{tikzcd}
 & \begin{pmatrix} B_1 & B_1 & B_3 \end{pmatrix} \arrow[rrr, "B_1\to B_3"] \arrow[ddr, "B_3\to 0"] & & & \begin{pmatrix} B_1 & B_1 & B_1 \end{pmatrix} \arrow[dr, "B_1\to 0"] & \\
\begin{pmatrix} B_1 & B_2 & B_3 \end{pmatrix} \arrow[ur, "B_1\to B_2"] \arrow[r, "B_2\to B_3"] \arrow[dr, "B_3\to 0"] & \begin{pmatrix} B_1 & B_2 & B_2 \end{pmatrix} \arrow[urrr, "B_1\to B_2"] \arrow[rr, "\sigma_2\to 0"] & & \begin{pmatrix} B_1 & 0 & 0 \end{pmatrix} \arrow[rr, "B_1\to 0"] & & \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \\
 & \begin{pmatrix} B_1 & B_2 & 0 \end{pmatrix} \arrow[urr, "B_2\to 0"] \arrow[r, "B_1\to B_2"] & \begin{pmatrix} B_1 & B_1 & 0 \end{pmatrix} \arrow[urrr, "B_1\to 0"] & & &
\end{tikzcd}$$

Family	$\begin{pmatrix} B_1 & B_2 & B_3 \end{pmatrix}$	$\begin{pmatrix} B_1 & B_2 & 0 \end{pmatrix}$	$\begin{pmatrix} B_1 & B_1 & B_3 \end{pmatrix}$ and $\begin{pmatrix} B_1 & B_2 & B_2 \end{pmatrix}$	$\begin{pmatrix} B_1 & B_1 & 0 \end{pmatrix}$	$\begin{pmatrix} B_1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} B_1 & B_1 & B_1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$
Label	#1	#2	#3 and #4	#5	#6	#7	#0
Codimension	0	1	2	3	4	5	9
Parameters	3	2	2	1	1	1	0
Orbit dim.	6	6	5	5	4	3	0
Orbit type	$\frac{SO(3)\times SO(3)}{\Delta(\mathbb{Z}_2\times\mathbb{Z}_2)}$	$\frac{SO(3)\times SO(3)}{\Delta(\mathbb{Z}_2\times\mathbb{Z}_2)}$	$\frac{SO(3)\times SO(3)}{\Delta(O(2))}$	$\frac{SO(3)\times SO(3)}{\Delta(O(2))}$	$\frac{\mathbb{S}^2\times\mathbb{S}^2}{\Delta(\mathbb{Z}_2)}$	$\Delta(SO(3))$	point
Rank	3	2	3	2	1	3	0
Minimal polynomial	0	$B_1B_2B_3$ $= \gamma$	$(B_1^2 - B_2^2)^2$ $\cdot (B_2^2 - B_3^2)^2$ $\cdot (B_3^2 - B_1^2)^2$ $=$ [see below]	$((B_1^2 - B_2^2)B_3)^2$ $+ ((B_2^2 - B_3^2)B_1)^2$ $+ ((B_3^2 - B_1^2)B_2)^2$ $= \alpha\beta - 9\gamma^2$	$(B_1B_2)^2$ $+ (B_2B_3)^2$ $+ (B_3B_1)^2$ $= \beta$	$\frac{1}{2}((B_1^2 - B_2^2)^2$ $+ (B_2^2 - B_3^2)^2$ $+ (B_3^2 - B_1^2)^2)$ $= \alpha^2 - 3\beta$	$B_1^2 + B_2^2 + B_3^2$ $= \alpha$

$$((B_1^2 - B_2^2)(B_2^2 - B_3^2)(B_3^2 - B_1^2))^2 = (\alpha\beta)^2 + 18\alpha\beta\gamma^2 - 4\beta^3 - 4\alpha^3\gamma^2 - 27\gamma^4.$$

Table 4.2: Stratification of 3 × 3 matrices

$B_1B_2B_3$ is invariant. The ring of invariant polynomials is $\mathbb{R}[\alpha, \beta, \gamma]$, where α, β, and γ are defined by

$$\begin{aligned}\alpha &= B_1^2 + B_2^2 + B_3^2,\\ \beta &= (B_1B_2)^2 + (B_2B_3)^2 + (B_3B_1)^2,\\ \gamma &= B_1B_2B_3.\end{aligned}$$

For example,

$$|B|^2 = 2(B_1^2 + B_2^2 + B_3^2) = 2\alpha.$$

Thus

$$|[B \,\boldsymbol{.}\, B]|^2 = 128((B_2B_3)^2 + (B_3B_1)^2 + (B_1B_2)^2) = 128\beta,$$

and

$$\langle B \cdot [B \,\boldsymbol{.}\, B]\rangle = -48B_1B_2B_3 = -48\gamma.$$

Thus the three functions

$$|B|^2 \,,\ |[B \,\boldsymbol{.}\, B]|^2 \,,\ \langle B \cdot [B \,\boldsymbol{.}\, B]\rangle$$

generate the ring of invariant functions.

The space of invariant quartics is a two-dimensional vector space, generated by α^2 and β. Consider now the invariant quartic function $|\langle B \odot B\rangle|^2$, where $\odot$ is the traceless symmetric product on $\Lambda^{2,+}$ defined in Section B.4. We compute

$$\begin{aligned}\langle B \odot B\rangle &= B_1^2\sigma^1 \odot \sigma^1 + B_2^2\sigma^2 \odot \sigma^2 + B_3^2\sigma^3 \odot \sigma^3\\ &= \tfrac{1}{3}\left((2B_1^2 - B_2^2 - B_3^2)\sigma^1 \otimes \sigma^1 + (2B_2^2 - B_3^2 - B_1^2)\sigma^2 \otimes \sigma^2 + (2B_3^2 - B_1^2 - B_2^2)\sigma^3 \otimes \sigma^3\right).\end{aligned}$$

Since $|\sigma^1 \otimes \sigma^2|^2 = 4$, we get

$$\begin{aligned}|\langle B \odot B\rangle|^2 &= \tfrac{4}{9}\left((2B_1^2 - B_2^2 - B_3^2)^2 + (2B_2^2 - B_3^2 - B_1^2)^2 + (2B_3^2 - B_1^2 - B_2^2)^2\right)\\ &= \tfrac{8}{3}\left(B_1^4 + B_2^4 + B_3^4 - (B_2B_3)^2 - (B_3B_1)^2 - (B_1B_2)^2\right)\\ &= \tfrac{4}{3}\left((B_2^2 - B_3^2)^2 + (B_3^2 - B_1^2)^2 + (B_1^2 - B_2^2)^2\right)\\ &= \tfrac{8}{3}(\alpha^2 - 3\beta).\end{aligned}$$

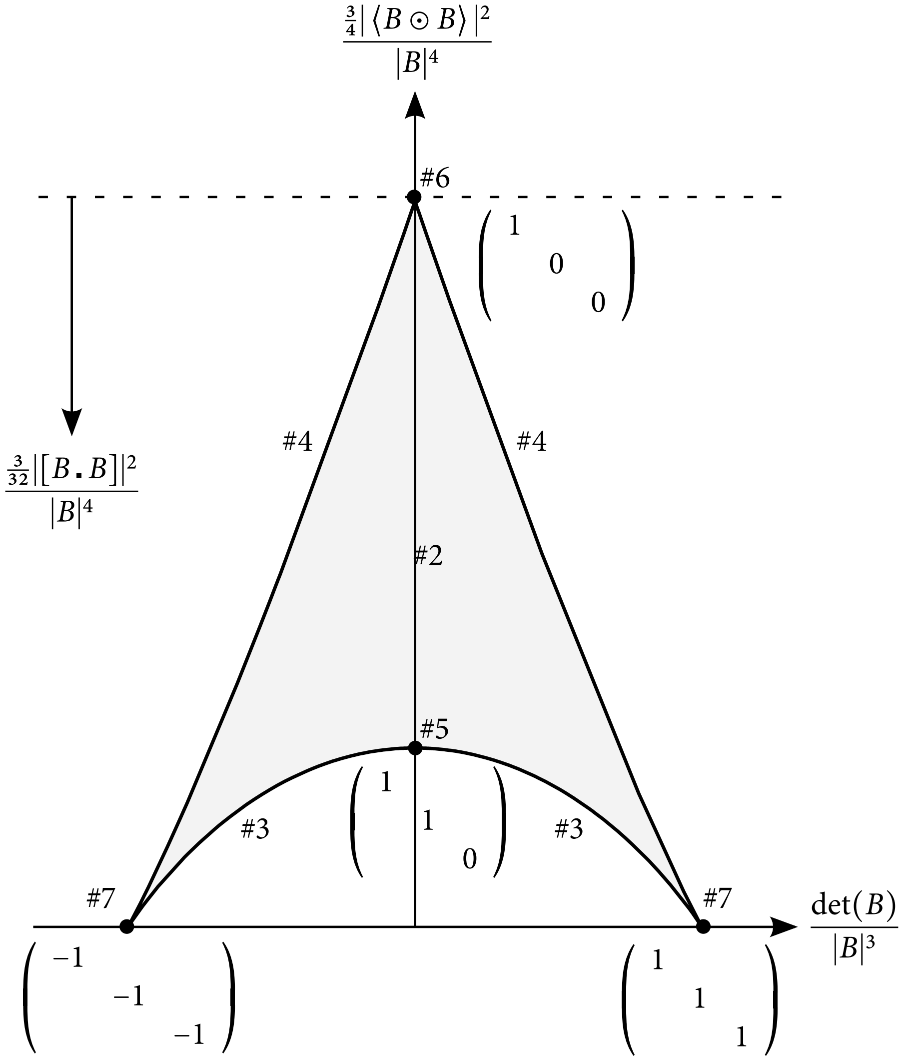
The quartics $|[B \,\boldsymbol{.}\, B]|^2$ and $|\langle B \odot B\rangle|^2$ are both positive semi-definite, and they vanish respectively on matrices of the form

$$\begin{pmatrix} B_1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \text{ and } \begin{pmatrix} B_1 & & \\ & B_1 & \\ & & B_1 \end{pmatrix}.$$

A routine computation proves

Theorem 4.1.1. *The space of positive semi-definite quartics on* $\Lambda^{2,+} \otimes \mathfrak{sp}(1)$ *is spanned by nonnegative combinations of* $|[B \,\boldsymbol{.}\, B]|^2$ *and* $|\langle B \odot B\rangle|^2$.

Figure 4-1: Rays in $\mathrm{SO}(3)\backslash M_{3\times 3}/\mathrm{SO}(3)$. The numbers label the strata described in Table 4.2 on page 39.



For example, $|B|^4 = \frac{3}{4}\left|\langle B\odot B\rangle\right|^2 + \frac{3}{32}\left|[B\,\mathbf{.}\,B]\right|^2$, and hence

$$|B| = \sqrt[4]{\tfrac{3}{4}\left|\langle B\odot B\rangle\right|^2 + \tfrac{3}{32}\left|[B\,\mathbf{.}\,B]\right|^2}.$$

We can now give a nice visualization of the space of (rays of) orbits in $M_{3\times 3}$. Consider the functions on $M_{3\times 3} - \{0\}$ given by

$$\frac{\frac{3}{4}\left|\langle B\odot B\rangle\right|^2}{|B|^4},\ \frac{\frac{3}{32}\left|[B\,\mathbf{.}\,B]\right|^2}{|B|^4},\ \frac{\langle B\cdot[B\,\mathbf{.}\,B]\rangle}{|B|^3}.$$

These functions are $\mathrm{SO}(3)\times\mathrm{SO}(3)$-invariant and constant along rays. The first two functions are complementary in that they sum to one. Moreover, they uniquely characterize each ray in $\mathrm{SO}(3)\backslash M_{3\times 3}/\mathrm{SO}(3)$. Plotting these functions, we get a planar region classifying the rays of orbits.

4.1.3 Completing the square

Introduce the function

$$\vartheta(s) := \begin{cases} \sqrt{\tfrac{1}{2}}s & \text{if } s \geq 0, \\ s & \text{if } s \leq 0, \end{cases}$$

so that

$$-s\sqrt{a^2+b^2} \leq -\vartheta(s)(|a|+|b|).$$

$$\begin{aligned}
& W^+ \cdot \langle B \odot B \rangle - \tfrac{1}{6} s \, |B|^2 - \tfrac{1}{32} \left|[B \,\raisebox{0.2ex}{.}\, B]\right|^2 \\
\leq\ & W^+ \cdot \langle B \odot B \rangle - \sqrt{\tfrac{1}{384}} \vartheta(s) \left(2\sqrt{2} \left|\langle B \odot B \rangle\right| + \left|[B \,.\, B]\right| \right) - \tfrac{1}{32} \left|[B \,.\, B]\right|^2 \\
=\ & \left(|W^+| - \sqrt{\tfrac{1}{48}} \vartheta(s) \right) \left|\langle B \odot B \rangle\right| - \tfrac{1}{32} \left(\left|[B \,.\, B]\right| + \sqrt{\tfrac{2}{3}} \vartheta(s) \right)^2 + \tfrac{1}{48} \vartheta(s)^2 \\
\leq\ & \left(|W^+| - \sqrt{\tfrac{1}{48}} \vartheta(s) \right) \left|\langle B \odot B \rangle\right| - \tfrac{1}{32} \left((1-\varepsilon) \left|[B \,.\, B]\right|^2 - \vartheta(s)^2/\varepsilon \right) \ \forall \varepsilon > 0.
\end{aligned}$$

In particular,

$$\|F_A\|_{L^2}^2 \leq 2\varepsilon_{\text{top}} + R \left(1 + \|\langle B \odot B \rangle\|_{L^1} \right) - \tfrac{1}{33} \|[B \,.\, B]\|_{L^2}^2 - \tfrac{1}{2} \|\nabla_A B\|_{L^2}^2 , \tag{4.1}$$

for some constant R depending only on curvature.

4.1.4 Rank one matrices

We now provide a geometric interpretation of (4.1) in terms of the distance to rank one matrices.

Define

$$Z := \left\{ B \in \mathfrak{sp}(1) \otimes \Lambda^{2,+} \,\middle|\, \operatorname{rank}(B) = 1 \right\} = \left\{ B \mid [B \,.\, B] = 0 \right\}.$$

According to Table 4.2 on page 39, this is a five-dimensional subset of $\mathbb{R}^9$ which is a cone on $(S^2 \times S^2)/\mathbb{Z}_2$. The codimension is four, so a generic B on a four-manifold will intersect Z at isolated points. Note that Z is the zero set of $\left|[B \,.\, B]\right|^2$. Assuming that $|B_1| \geq |B_2| \geq |B_3|$, the distance from B to Z is given by

$$\operatorname{dist}(B, Z)^2 = 2(B_2^2 + B_3^2),$$

where the factor of two arises from the fact that our basis vectors have norm two. We have the following identity:

$$(B_1B_2)^2 + (B_2B_3)^2 + (B_3B_1)^2 \leq (B_1^2 + B_2^2 + B_3^2)(B_2^2 + B_3^2) \leq 2\left((B_1B_2)^2 + (B_2B_3)^2 + (B_3B_1)^2 \right).$$

This leads to

$$\left|[B \,.\, B]\right|^2 \leq 32 \, |B|^2 \operatorname{dist}(B, Z)^2 \leq 2 \left|[B \,.\, B]\right|^2 .$$

In particular,

$$C \left|\langle B \odot B \rangle\right|^2 - \tfrac{1}{16} \left|[B \,.\, B]\right|^2 \leq |B|^2 \left(C - \operatorname{dist}(B, Z)^2 \right).$$

In order to lose control over $\|F_A\|$, it's necessary for the L^2-density of B to accumulate in a region within distance $\sqrt{C}$ of Z, where C is some distance determined by the curvature.

4.2 Unique continuation

We have the following simple but powerful corollary of unique continuation for ASD connections.

Theorem 4.2.1. *Let X be a simply-connected oriented Riemannian four-manifold, and let $P \to X$ be an* $\mathrm{SU}(2)$ *principal bundle. If $A \in \mathcal{A}_P$ and $B \in \Omega^{2,+}(M; \mathfrak{g}_P)$ satisfy*

$$F_A^+ = 0,$$
$$d_A^* B = 0,$$
$$A \text{ is irreducible},$$

then $B = 0$.

Since $F_A^+ = 0$, by Table 4.1 on page 38, B has at most rank one. Let Z^c denote the complement of the zero set of B. By unique continuation of the elliptic equation $d_A^* B = 0$, Z^c is either empty or dense. On Z^c write $B = \xi \otimes \omega$ for $\xi \in \Omega^0(Z^c; \mathfrak{g}_P)$ with $\langle \xi, \xi \rangle = 1$, and $\omega \in \Omega^{2,+}(Z^c)$. We compute

$$0 = -d_A^*(\xi \otimes \omega) = \iota_i \nabla_{A,i}(\xi \otimes \omega) = (\nabla_{A,i}\xi) \otimes a_i \omega - \xi \otimes (d^* \omega) = d_A \xi \,\centerdot\, \omega - \xi \otimes d^* \omega.$$

Taking the inner product with ξ and using the consequence of $\langle \xi, \xi \rangle = 1$ that $\langle \xi, d_A \xi \rangle = 0$, we get $d^* \omega = 0$. It follows that $d_A \xi \,\centerdot\, \omega = 0$. Note that this pairing is definite, since in components $(v_0, \vec{v}) \,\centerdot\, \vec{\omega} = (-\vec{v} \cdot \vec{w}, v_0 \vec{\omega} - \vec{v} \times \vec{\omega})$. Since ω is nowhere zero along Z^c, we must have $d_A \xi = 0$ along Z^c. Therefore, A is reducible along Z^c. However according to [DK97] Lemma (4.3.21) p. 150, A is irreducible along Z^c. This is a contradiction unless Z^c is empty. Therefore $Z = X$, so B is identically zero.

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Chapter 5

Perturbing the metric

5.1 Metrics and conformal structures in four dimensions

5.1.1 Fundamentals

Let (V, g_0) be an oriented vector space with a fixed "base" Riemannian inner product $g_0 \in \mathrm{Met}(V)$. The purpose of this section is to express an arbitrary Riemannian inner product g in terms of the "base" inner product g_0 by using representation theory of the g_0-orthogonal group. This will allow us to express certain operators associated with g in terms of the original g_0 operators in the subsequent section.

Remark. Unless otherwise noted, all metric-dependent operators and spaces such as Hodge star $\star$, inner product $\cdot$, volume form $d\mathrm{vol}$, and self-dual/anti-self-dual forms $\Lambda^{2,\pm}V^*$ implicitly refer the base metric g_0. Operators and spaces determined by g are denoted by a subscript, i.e. $\Lambda^{2,\pm}_g V^*$.

Define $\mathrm{Hom}(\Lambda^{2,-}V^*, \Lambda^{2,+}V^*)_{<1}$ to be the subset of linear maps with operator norm less than one.

Theorem 5.1.1. *Any Riemannian inner product $g \in \mathrm{Met}(V)$ corresponds (relative to g_0) to a unique pair*

$$(\eta, m) \in \mathbb{R} \times \mathrm{Hom}(\Lambda^{2,-}V^*, \Lambda^{2,+}V^*)_{<1}$$

such that $d\mathrm{vol}_g = e^{4\eta} d\mathrm{vol}$, and the graph of m inside of $\Lambda^2 V^*$ is $\Lambda^{2,-}_g V^*$. Furthermore, $\Lambda^{2,+}_g V^*$ is the graph of the g_0-adjoint map $m^* \in \mathrm{Hom}(\Lambda^{2,+}V^*, \Lambda^{2,-}V^*)$.

Proof. We will need the $\mathrm{SO}(4, g_0)$-equivariant map

$$\mu : \mathrm{Sym}^2(V^*) \to \mathrm{Hom}(\Lambda^{2,-}V^*, \Lambda^{2,+}V^*),$$
$$\mu(\alpha \otimes \beta) := \big(\omega \mapsto (\star(\omega \wedge \alpha) \wedge \beta)^+\big).$$

Explicitly, if $\{e^0, e^1, e^2, e^3\}$ is an oriented g_0-orthonormal coframe for V^*, then with respect to the bases

$$\left\{e^0 \wedge e^1 - e^2 \wedge e^3, e^0 \wedge e^2 - e^3 \wedge e^1, e^0 \wedge e^3 - e^1 \wedge e^2\right\} \text{ for } \Lambda^{2,-}V^*,$$
$$\left\{e^0 \wedge e^1 + e^2 \wedge e^3, e^0 \wedge e^2 + e^3 \wedge e^1, e^0 \wedge e^3 + e^1 \wedge e^2\right\} \text{ for } \Lambda^{2,+}V^*,$$

we have the following table of matrices for possible inputs. (The lower triangle is redundant because of symmetry.)

$$
2\mu\begin{pmatrix} e^0\otimes e^0 & e^0\otimes e^1 & e^0\otimes e^2 & e^0\otimes e^3 \\ & e^1\otimes e^1 & e^1\otimes e^2 & e^1\otimes e^3 \\ & & e^2\otimes e^2 & e^2\otimes e^3 \\ & & & e^3\otimes e^3 \end{pmatrix}
$$

$$
= \begin{array}{cccc}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
 & \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
 & & \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
 & & & \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{array}.
$$

For example,

$$
\mu(e^0\otimes e^1)(e^0\wedge e^2 - e^3\wedge e^1) = (\star(e^0\wedge e^1\wedge e^3)\wedge e^1)^+ = \tfrac{1}{2}\left(e^0\wedge e^3 + e^1\wedge e^2\right),
$$

and this answer corresponds to the entry "1" in the second matrix above.

Note that $\ker\mu$ is the span of $g_0 \in \mathrm{Sym}^2(V^*)$, and when μ is restricted to the g_0-traceless tensors $\mathrm{Sym}^2_0(V^*)$, it becomes an isomorphism

$$
\mu : \mathrm{Sym}^2_0(V^*) \xrightarrow{\cong} \mathrm{Hom}(\Lambda^{2,-}V^*, \Lambda^{2,+}V^*).
$$

Using the canonical isomorphism $V^*\otimes V^* \cong \mathrm{Hom}(V, V^*)$, we adopt the viewpoint that $g_0, g \in \mathrm{Hom}(V, V^*)$. Thus $g_0^{-1}g \in \mathrm{End}(V)$ defines a g_0-symmetric endomorphism with positive eigenvalues. Therefore, $\ln(g_0^{-1}g) \in \mathrm{End}(V)$ exists, and is g_0-symmetric.

Consider

$$
g_0 \ln(g_0^{-1}g) \in \mathrm{Hom}(V, V^*) \cong V^*\otimes V^*.
$$

In particular,

$$
g_0 \ln(g_0^{-1}g) \in \mathrm{Sym}^2(V^*),
$$

so

$$
\mu(g_0 \ln(g_0^{-1}g)) \in \mathrm{Hom}(\Lambda^{2,-}V^*, \Lambda^{2,+}V^*).
$$

The corresponding matrix has a singular value decomposition. Any odd function $f : \mathbb{R} \to \mathbb{R}$ when applied to singular values induces a bi-invariant function

$$
f : \mathrm{Hom}(\Lambda^{2,-}V^*, \Lambda^{2,+}V^*) \longrightarrow \mathrm{Hom}(\Lambda^{2,-}V^*, \Lambda^{2,+}V^*).
$$

We define

$$\begin{aligned} m &:= \tanh(\mu(\tfrac{1}{2} g_0 \ln(g_0^{-1} g))) \in \mathrm{Hom}(\Lambda^{2,-} V^*, \Lambda^{2,+} V^*)_{<1}, \\ \eta &:= \tfrac{1}{8} \mathrm{tr}(\ln(g_0^{-1} g)). \end{aligned}$$

The metric g is determined by the inverse construction

$$g = e^{2\eta} g_0 \exp\left(2 g_0^{-1} \mu^{-1}(\tanh^{-1} m)\right). \tag{5.1}$$

These constructions give the desired bijection

$$\begin{aligned} \mathbb{R} \times \mathrm{Hom}(\Lambda^{2,-} V^*, \Lambda^{2,+} V^*)_{<1} &\longleftrightarrow \mathrm{Met}(V), \\ (\eta, m) &\longleftrightarrow g. \end{aligned}$$

It remains to show that η and m satisfy the claimed properties.

We start by showing that $d\mathrm{vol}_g = e^{4\eta} d\mathrm{vol}$. We use the formula $d\mathrm{vol}_g = \sqrt{\det(g_0^{-1} g)} d\mathrm{vol}$ to compute

$$d\mathrm{vol}_g = \sqrt{\det(e^{2\eta} I) \det(\exp(g_0^{-1} \mu^{-1}(\cdots)))} d\mathrm{vol}.$$

The first factor yields $\sqrt{\det(e^{2\eta} I)} = e^{4\eta}$, while the second factor $\sqrt{\det(\exp(g_0^{-1} \mu^{-1}(\cdots)))}$ is one because $g_0^{-1} \mu^{-1}(\cdots)$ is traceless.

Finally, we show that $\Lambda_g^{2,-} V^*$ is the graph of m inside of $\Lambda^2 V^*$. Since $g_0^{-1} g$ is g_0-symmetric, we may choose an orthonormal coframe $\{e^0, e^1, e^2, e^3\}$ such that

$$\begin{aligned} g_0^{-1} g &= \begin{pmatrix} \eta_0^2 & 0 & 0 & 0 \\ 0 & \eta_1^2 & 0 & 0 \\ 0 & 0 & \eta_2^2 & 0 \\ 0 & 0 & 0 & \eta_3^2 \end{pmatrix}, \\ \mu(\tfrac{1}{2} g_0 \ln(g_0^{-1} g)) &= \begin{pmatrix} \ln(\eta_0 \eta_1 / \eta_2 \eta_3) & 0 & 0 \\ 0 & \ln(\eta_0 \eta_2 / \eta_3 \eta_1) & 0 \\ 0 & 0 & \ln(\eta_0 \eta_3 / \eta_1 \eta_2) \end{pmatrix}, \\ m &= \begin{pmatrix} \dfrac{\eta_0 \eta_1 - \eta_2 \eta_3}{\eta_0 \eta_1 + \eta_2 \eta_3} & 0 & 0 \\ 0 & \dfrac{\eta_0 \eta_2 - \eta_3 \eta_1}{\eta_0 \eta_2 + \eta_3 \eta_1} & 0 \\ 0 & 0 & \dfrac{\eta_0 \eta_3 - \eta_1 \eta_2}{\eta_0 \eta_3 + \eta_1 \eta_2} \end{pmatrix}. \end{aligned}$$

(Compare with [DS89] Lemma 2.3.)

Note that $\{\eta_0 e^0, \eta_1 e^1 \eta_2 e^2, \eta_3 e^3\}$ is a g-orthonormal coframe. Thus

$$\begin{aligned}
\Lambda^{2,-}_g V^* &= \operatorname{span}\left\{\eta_0\eta_1 e^0 \wedge e^1 - \eta_2\eta_3 e^2 \wedge e^3, \eta_0\eta_2 e^0 \wedge e^2 - \eta_3\eta_1 e^3 \wedge e^1, \eta_0\eta_3 e^0 \wedge e^3 - \eta_1\eta_2 e^1 \wedge e^2\right\} \\
&= \operatorname{span}\left\{2\frac{\eta_0\eta_1 e^0 \wedge e^1 - \eta_2\eta_3 e^2 \wedge e^3}{\eta_0\eta_1 + \eta_2\eta_3}, \cdots\right\} \\
&= \operatorname{span}\left\{e^0 \wedge e^1 - e^2 \wedge e^3 + \frac{\eta_0\eta_1 - \eta_2\eta_3}{\eta_0\eta_1 + \eta_2\eta_3}(e^0 \wedge e^1 + e^2 \wedge e^3), \cdots\right\} \\
&= \operatorname{span}\left\{(1+m)\left(e^0 \wedge e^1 - e^2 \wedge e^3\right), \cdots\right\} \\
&= \operatorname{graph}(m).
\end{aligned}$$

The same type of computation applies to show

$$\begin{aligned}
\Lambda^{2,+}_g V^* &= \operatorname{span}\left\{e^0 \wedge e^1 + e^2 \wedge e^3 + \frac{\eta_0\eta_1 - \eta_2\eta_3}{\eta_0\eta_1 + \eta_2\eta_3}(e^0 \wedge e^1 - e^2 \wedge e^3), \cdots\right\} \\
&= \operatorname{graph}(m^*).
\end{aligned}$$

□

It will be helpful to identify $\Lambda^{2,+}V^*$ with $\Lambda^{2,+}_g V^*$. In light of Theorem 5.1.1, the most obvious identification is the projection

$$1 \oplus m^* : \Lambda^{2,+}V^* \longrightarrow \Lambda^{2,+}_g V^* \subset \Lambda^{2,+}V^* \oplus \Lambda^{2,-}V^*.$$

However, the most geometrically relevant identification is isometric:

Theorem 5.1.2. *The map $e^{2\eta}(1 \oplus m^*)(1 - mm^*)^{-1/2}$ is an isometry from $\Lambda^{2,+}V^*$ to $\Lambda^{2,+}_g V^*$.*

Proof. We will verify this lemma by computation. Some care is required since the decomposition $\Lambda^2 V^* = \Lambda^{2,+}V^* \oplus \Lambda^{2,-}V^*$ is not g-orthogonal. To compute the g-inner product, we instead employ the formulas

$$\begin{aligned}
\tilde\omega_1 \cdot_g \tilde\omega_2 &= \tilde\omega_1 \wedge \tilde\omega_2 \, d\mathrm{vol}_g^{-1} \text{ for all } \tilde\omega_1, \tilde\omega_2 \in \Lambda^{2,+}_g V^*, \\
\omega_1^+ \cdot \omega_2^+ &= \omega_1^+ \wedge \omega_2^+ d\mathrm{vol}^{-1} \text{ for all } \omega_1^+, \omega_2^+ \in \Lambda^{2,+}V^*, \\
-\omega_1^- \cdot \omega_2^- &= \omega_1^- \wedge \omega_2^- d\mathrm{vol}^{-1} \text{ for all } \omega_1^-, \omega_2^- \in \Lambda^{2,-}V^*.
\end{aligned}$$

We compute

$$
\begin{aligned}
&\left(e^{2\eta}(1\oplus m^*)(1-mm^*)^{-1/2}\omega_1\right)\cdot_g\left(e^{2\eta}(1\oplus m^*)(1-mm^*)^{-1/2}\omega_2\right)\\
&=e^{4\eta}\left((1\oplus m^*)(1-mm^*)^{-1/2}\omega_1\right)\wedge\left((1\oplus m^*)(1-mm^*)^{-1/2}\omega_2\right)\,d\mathrm{vol}_g^{-1}\\
&=\frac{((1-mm^*)^{-1/2}\omega_1)\wedge((1-mm^*)^{-1/2}\omega_2)+(m^*(1-mm^*)^{-1/2}\omega_1)\wedge(m^*(1-mm^*)^{-1/2}\omega_2)}{d\mathrm{vol}}\\
&=\left((1-mm^*)^{-1/2}\omega_1\right)\cdot\left((1-mm^*)^{-1/2}\omega_2\right)-\left(m^*(1-mm^*)^{-1/2}\omega_1\right)\cdot\left(m^*(1-mm^*)^{-1/2}\omega_2\right)\\
&=\omega_1\cdot\left((1-mm^*)^{-1}-(1-mm^*)^{-1/2}mm^*(1-mm^*)^{-1/2}\right)\omega_2\\
&=\omega_1\cdot\omega_2.
\end{aligned}
$$

□

5.1.2 Operators associated with a perturbed metric

Theorem 5.1.3. *If $\omega\in\Lambda^2V^*$ decomposes under g_0 as*

$$\omega=\omega^+\oplus\omega^-\in\Lambda^{2,+}V^*\oplus\Lambda^{2,-}V^*,$$

then the g-decomposition is given by the identity

$$
\begin{aligned}
\omega^+\oplus\omega^-&=\overbrace{(1\oplus m^*)\left(1-mm^*\right)^{-1}(\omega^+-m\omega^-)}^{\Lambda_g^{2,+}V^*}+\overbrace{(1\oplus m)\left(1-m^*m\right)^{-1}(-m^*\omega^++\omega^-)}^{\Lambda_g^{2,-}V^*}.\\
&=e^{2\eta}(1\oplus m^*)(1-mm^*)^{-1/2}\left(e^{-2\eta}\left(1-mm^*\right)^{-1/2}(\omega^+-m\omega^-)\right)\\
&\quad+e^{2\eta}(1\oplus m)\left(1-m^*m\right)^{-1/2}\left(e^{-2\eta}\left(1-m^*m\right)^{-1/2}(-m^*\omega^++\omega^-)\right).
\end{aligned}
$$

Proof. The second equality follows directly from the first. By Theorem 5.1.1, the images of $1\oplus m^*:\Lambda^{2,+}V^*\to\Lambda^2V^*$ and $1\oplus m:\Lambda^{2,-}V^*\to\Lambda^2V^*$ are manifestly g-self-dual and g-anti-self-dual respectively. This shows that both terms belong to the appropriate subspaces. It remains to show that the right hand side is indeed the left hand side.

Collecting on ω^+ and ω^-, the right hand side is

$$
\begin{aligned}
&\left((1\oplus m^*)\left(1-mm^*\right)^{-1}-(1\oplus m)\left(1-m^*m\right)^{-1}m^*\right)\omega^++\\
&\qquad\qquad+\left((1\oplus m)\left(1-m^*m\right)^{-1}-(1\oplus m^*)\left(1-mm^*\right)^{-1}m\right)\omega^-.
\end{aligned}
$$

The result then follows from identities in the spirit of

$$\left(1-m^*m\right)^{-1}m^*=m^*\left(1-mm^*\right)^{-1}.$$

□

Theorem 5.1.4. *For $\omega_1, \omega_2 \in \Lambda^{2,+}V^*$, let $\tilde{\omega}_i = e^{2\eta}(1 \oplus m^*)(1 - mm^*)^{-1/2}\omega_i$. Then*

$$e^{2\eta}(1 \oplus m^*)(1 - mm^*)^{-1/2}\left(\omega_1 \mathbin{\boldsymbol{.}} \omega_2\right) = \tilde{\omega}_1 \mathbin{\boldsymbol{.}} \tilde{\omega}_2.$$

Proof. The restrictions of the products $\boldsymbol{.}$ to $\Lambda^{2,+}$ and $\boldsymbol{.}_g$ to $\Lambda^{2,+}_g$ are both equivalent to the same multiple of the standard cross product on $\mathbb{R}^3$. In particular, they are equivariant under isometry. The expression in this theorem is the formula for this equivariance. □

Theorem 5.1.5. *If $\omega \in \Lambda^3 V^*$, then*

$$\star_g \omega = e^{-2\eta} g_m g_0^{-1}(\star\omega),$$

where $g_m := e^{-2\eta}g$ is the metric determined by the pair $(0, m) \in \mathbb{R} \times \mathrm{Hom}(\Lambda^{2,-}V^*, \Lambda^{2,+}V^*)_{<1}$.

Proof. Dual to $d\mathrm{vol}_g \in \Lambda^4 V^*$ is $d\mathrm{vol}_g^{-1} \in \Lambda^4 V$ such that under the natural pairing,

$$d\mathrm{vol}_g \cdot d\mathrm{vol}_g^{-1} = 1.$$

One algorithm for computing $\star_g$ is to first form the contraction

$$\omega \cdot d\mathrm{vol}_g^{-1} \in \Lambda^3 V^* \otimes \Lambda^4 V \cong \Lambda^1 V,$$

and then apply g:

$$\star_g \omega := g(\omega \cdot d\mathrm{vol}_g^{-1}) \in \Lambda^1 V^*.$$

To obtain the desired form, we rewrite this as

$$\begin{aligned}
\star_g \omega &= e^{2\eta} g_m(\omega \cdot e^{-4\eta} d\mathrm{vol}^{-1}) \\
&= e^{-2\eta} g_m \left(g_0^{-1} g_0\right)\left(\omega \cdot d\mathrm{vol}^{-1}\right) \\
&= e^{-2\eta} g_m g_0^{-1}(\star\omega).
\end{aligned}$$

□

5.2 Perturbing the metric of the local model

Theorem 5.2.1. *Let X be an oriented Riemannian four-manifold with metric g_0. Consider any pair*

$$\eta \in \Omega^0(X; \mathbb{R}) \text{ and } m \in \Omega^0(X; \mathrm{Hom}(\Lambda^{2,-}V^*, \Lambda^{2,+}V^*)_{<1}).$$

Recall that by 5.1.1, $e^{2\eta}$ determines a conformal factor, m determines a change of conformal structure, and together these uniquely determine any new metric g.

For $B \in \Omega^{2,+}_{g_0}(\mathfrak{g}_P)$, we define

$$\tilde{B} := (1 \oplus m^*)(1 - mm^*)^{-1/2} B,$$

so that $\tilde{B} \in \Omega^{2,+}_g(\mathfrak{g}_P)$ by 5.1.2. Then

$$\begin{aligned} &\mathrm{VW}(e^{-1\eta}C, A, e^{+1\eta}\tilde{B}, \eta, m) \\ &= \begin{pmatrix} e^{-1\eta}\left(g_m g_0^{-1}(-\star\, d_A\tilde{B}) + d_A C - C\, d\eta - g_m g_0^{-1} d\eta \,\mathbf{.}\, \tilde{B}\right) \\ e^{0\eta}(1 \oplus m^*)(1 - mm^*)^{-1/2}\left((1 - mm^*)^{-1/2}(F_A^+ - mF_A^-) + \frac{1}{8}[B \,\mathbf{.}\, B] + \frac{1}{2}[B, C]\right) \end{pmatrix}. \end{aligned} \tag{5.2}$$

Furthermore, to first order in $|\eta| + |m|$,

$$\begin{aligned} &\mathrm{VW}(e^{-1\eta}C, A, e^{+1\eta}\tilde{B}, \eta, m) \\ &= \begin{pmatrix} e^{-\eta}\left(d_A^* B + d_A C - C\, d\eta + 2\mu^{-1}(m) g_0^{-1} d_A^* B - d\eta \,\mathbf{.}\, B - d_A^* m^* B\right)\big) + O(|m| + |\nabla m| + |d\eta|)^2 \\ (1 \oplus m^*)\left(F_A^+ + \frac{1}{8}[B \,\mathbf{.}\, B] + \frac{1}{2}[B, C] - mF_A^-\right) + O(|m|^2) \end{pmatrix}. \end{aligned} \tag{5.3}$$

Proof. First we verify (5.2), starting with the $\Omega^1(\mathfrak{g}_P)$ component. Two of these terms follow from $d_A(e^{-\eta}C) = e^{-\eta}(d_A C - C\, d\eta)$. For the remaining terms, note that on $\Omega^{2,+}_g(\mathfrak{g}_P)$,

$$d_A^{*_g} = -\star_g d_A \star_g = -\star_g d_A = -e^{-2\eta} g_m g_0^{-1} \star d_A,$$

where the last equality follows from Theorem 5.1.5. Thus

$$\begin{aligned} d_A^{*_g}(e^{\eta}\tilde{B}) &= -e^{-\eta} g_m g_0^{-1} \star (d\eta \wedge \tilde{B} + d_A\tilde{B}) \\ &= e^{-\eta} g_m g_0^{-1}(-d\eta \,\mathbf{.}\, \tilde{B} - \star d_A\tilde{B}). \end{aligned}$$

This accounts for the $\Omega^1(\mathfrak{g}_P)$ component.

To verify the $\Omega^{2,+}_g(\mathfrak{g}_P)$ component of (5.2), first note that the terms involving F_A follow from Theorem 5.1.3. Next, taking care with the slight modification of the conformal weight in Theorem 5.1.4, we get

$$\left[\tilde{B} \,\mathbf{.}\, \tilde{B}\right] = (1 \oplus m^*)(1 - mm^*)^{-1/2}[B \,\mathbf{.}\, B].$$

Finally, noting that $[e^{\eta}B, e^{-\eta}C] = [B, C]$, this accounts for the $\Omega^{2,+}_g(\mathfrak{g}_P)$ component of (5.2).

Now we verify (5.3). The $\Omega^{2,+}_g(\mathfrak{g}_P)$ component is clear, but the $\Omega^1(\mathfrak{g}_P)$ component requires some computation. To first order in m,

$$d_A\tilde{B} = d_A(1 \oplus m^*)B + O(|m|^2) = d_A B + d_A m^* B + O(|m|^2).$$

Thus

$$-\star\, d_A\tilde{B} = d_A^*(B - m^*B) + O(|m|^2).$$

From (5.1), we have

$$g_m = g_0 + 2\mu^{-1}(m) + O(|m|^2),$$

so

$$g_m g_0^{-1}(-\star\, d_A\tilde{B}) = d_A^* B - d_A^* m^* B + 2\mu^{-1}(m) g_0^{-1} d_A^* B + O(|m|^2 + |m|\,|\nabla m|).$$

Finally,

$$\begin{aligned}-g_m g_0^{-1} d\eta \,.\, \tilde{B} &= -d\eta \,.\, \tilde{B} + O(|m|\,|d\eta|)\\ &= -d\eta \,.\, B + O(|m|\,|d\eta|).\end{aligned}$$

This accounts for all the terms appearing in the $\Omega^1(\mathfrak{g}_P)$ component of (5.3). $\square$

Now consider now a perturbation $(C+c, A+a, B+b)$ together with the first order metric perturbation. (All second order terms in $|\eta| + |m|$ are implicitly set to zero.)

$$\begin{aligned}&\mathrm{VW}(C+c, A+a, B+b, \eta, m) \qquad (5.4)\\ &= \mathrm{VW}(C+c, A+a, B+b, 0, 0)+\\ &\quad -(C+c)\, d\eta + 2\mu^{-1}(m) g_0^{-1} d^*_{A+a}(B+b) - d\eta \,.\, (B+b) - d^*_{A+a} m^*(B+b) \oplus -mF^-_{A+a}\\ &= \mathrm{VW}(C+c, A+a, B+b, 0, 0) + (\mathrm{VW}(C, A, B, \eta, m) - \mathrm{VW}(C, A, B, 0, 0)) +\\ &\quad -c\, d\eta + 2\mu^{-1}(m) g_0^{-1}\left(d^*_A b - [a \,.\, B] - [a \,.\, b]\right) - d\eta \,.\, b - d^*_A m^* b - [a \,.\, m^* B] - [a \,.\, m^* b]\\ &\quad \oplus -m\left(d^-_A a + \tfrac{1}{2}[a \wedge a]^-\right).\end{aligned}$$

We will focus on the case of an ASD instanton when the conformal structure is perturbed. That is, we set $C = 0$, $B = 0$, $\eta = 0$, $d^+_A a = 0$, $d^*_A b = 0$. This reduces to

$$-2\mu^{-1}(m) g_0^{-1}[a \,.\, b] - d^*_A m^* b - [a \,.\, m^* b] \oplus -\tfrac{1}{2} m\, [a \wedge a]^-.$$

The quadratic model for this situation becomes

$$\left\langle \left(\left(1 - 2\mu^{-1}(m) g_0^{-1}\right)[a \,.\, b] - [a \,.\, m^* b]\right) \cdot \hat{a} \right\rangle = 0, \tag{5.5}$$

$$\left\langle \left(\tfrac{1}{2}[a \wedge a]^+ + \tfrac{1}{8}[b \,.\, b] - m\left(F^-_A + \tfrac{1}{2}[a \wedge a]^-\right)\right) \cdot \hat{b} \right\rangle = 0, \tag{5.6}$$

for all $\hat{a}$ and $\hat{b}$ in the cokernel. The important thing to note about these formulas is that (5.5) consists purely of cross-terms between a and b, while (5.6) contains no cross-terms.

Chapter 6

An abelian solution on hyperbolic space

Fundamental to our understanding of the ASD equations are the model solutions on flat $\mathbb{R}^4$. In this section, we construct a $\mathrm{U}(1)$ solution of finite energy on hyperbolic space. This solution is simply a scalar-valued harmonic self-dual two-form.

Although this example lives on a non-compact manifold, it allows us to explicitly verify many of our computations, and examine how the terms behave.

6.1 Geometry of hyperbolic space

Consider a family of hypersurfaces parameterized by t such that the metric is of the form $g = dt^2 + g(t)$, and $g(t)$ is the hypersurface metric. The normal curvature is $N_{ij} = -\frac{1}{2}(\ln g)'_{ij}$. If we take an orthonormal frame on a hypersurface and parallel transport it along the hypersurface normals, then the connection matrices take the form

$$
\begin{aligned}
\Gamma_0 &= 0 \\
\Gamma_i &= \left(\begin{array}{cccc} 0 & N_{i1} & N_{i2} & N_{i3} \\ -N_{i1} & & & \\ -N_{i2} & & \Gamma_i^{\parallel} & \\ -N_{i3} & & & \end{array} \right).
\end{aligned}
$$

The curvature components are determined in terms of the curvature of the hypersurface by

$$
\begin{aligned}
R_{0j0\ell} &= \dot{N}_{j\ell} - N_{jm}N_{m\ell}, \\
R_{0jk\ell} &= \dot{\Gamma}_{jk\ell} - N_{jm}\Gamma_{mk\ell}, \\
R_{ijk\ell} &= R^{\parallel}_{ijk\ell} - (N_{ik}N_{j\ell} - N_{i\ell}N_{jk}).
\end{aligned}
$$

As a sanity check, we can decompose flat Euclidean $\mathbb{R}^4$ along concentric three-spheres. In this case, we may take $\Gamma_{ijk} = -r^{-1}\varepsilon_{ijk}$, $N_{ij} = -r^{-1}\delta_{ij}$, $R^{\parallel}_{ijk\ell} = r^{-2}(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk})$, and verify that each curvature component vanishes.

Along $\mathbb{S}^4$, we parameterize a cylinder $\mathbb{S}^3 \times [0, \pi]$ with respect to the distance t. The radius of the $\mathbb{S}^3$ at time t is $r = \sin t$. The volume of $\mathbb{S}^3$ is $2\pi^2 r^3$. Normal curvature is $N_{ij} = (-r^{-1} \cos t) = (-\cot t)\delta_{ij}$. Upon replacing these trig functions by hyperbolic functions, we get hyperbolic space $\mathbb{H}^4$.

On $\mathbb{H}^4$, choose the orthonormal coframe

$$
\begin{aligned}
(e^0, e^1, e^2, e^3) := \Bigg(& \frac{1}{t}(t dt), \\
& \frac{\sinh t}{t} \frac{1}{t}(-x^1 dx^0 + x^0 dx^1 - x^3 dx^2 + x^2 dx^3), \\
& \frac{\sinh t}{t} \frac{1}{t}(-x^2 dx^0 + x^3 dx^1 + x^0 dx^2 - x^1 dx^3), \\
& \frac{\sinh t}{t} \frac{1}{t}(-x^3 dx^0 - x^2 dx^1 + x^1 dx^2 + x^0 dx^3)\Bigg).
\end{aligned}
$$

The volume form is

$$
e^{0123} = \left(\frac{\sinh t}{t}\right)^3 dx^{0123}.
$$

Although the $\{e^i\}$ are singular at the origin, the self-dual two-forms $e^{0i} + e^{(i+1)(i+2)}$ agree with $dx^{0i} + dx^{(i+1)(i+2)}$ to second order.

6.2 The solution

We set

$$
B = \frac{e^{01} + e^{23}}{\cosh^4(t/2)} \tag{6.1}
$$

and show that B is harmonic with finite L^2 energy. It has a primitive given by

$$
B = d\left(\frac{\sinh(t)\, e^1}{2\cosh^4(t/2)}\right).
$$

The norm of B over a ball of radius d is $\|B\|^2 = 16\pi^2 \tanh^4\left(\frac{t}{2}\right)$, since

$$
\begin{aligned}
B \cdot B &= \frac{2}{\cosh^8(t/2)}, \\
B \cdot B\, d\mathrm{vol}_{\mathbb{H}^4} &= \frac{2\sinh^3(t)}{\cosh^8(t/2)} dt \wedge d\mathrm{vol}_{\mathbb{S}^3} \\
\int B \cdot B\, d\mathrm{vol}_{\mathbb{H}^4} &= 4\pi^2 \int \frac{\sinh^3(t)}{\cosh^8(t/2)} dt = 16\pi^2 \tanh^4(t/2).
\end{aligned}
$$

We will now compute $\varepsilon_{\mathrm{an}}$.

$$
\nabla B = \left(-2\mathrm{sech}^4\left(\frac{t}{2}\right)\tanh\left(\frac{t}{2}\right)\sigma^1, 0, -\mathrm{sech}^4\left(\frac{t}{2}\right)\tanh\left(\frac{t}{2}\right)\sigma^3, \mathrm{sech}^4\left(\frac{t}{2}\right)\tanh\left(\frac{t}{2}\right)\sigma^2\right).
$$

The norm of ∇B is $\|\nabla B\|^2 = 64\pi^2 \tanh^6\left(\frac{t}{2}\right)$. Therefore,

$$\varepsilon_{\mathrm{an}} = \frac{1}{4}\|\nabla B\|^2 + \frac{s}{12}\|B\|^2 = 16\pi^2 \tanh^4\left(\frac{t}{2}\right)\left(\tanh^2\left(\frac{t}{2}\right) - 1\right) = -16\pi^2 \mathrm{sech}^2\left(\frac{t}{2}\right)\tanh^4\left(\frac{t}{2}\right).$$

For the topological energy,

$$\begin{aligned}
B^{\|} &= \frac{e^{23}}{\cosh^4(t/2)},\\
d\star B^{\|} &= \frac{2}{\sinh(t)}B^{\|}\\
\rho_{\mathrm{D}}(N)B^{\|} &= \frac{-2\cosh(t)}{\sinh(t)}B^{\|}\\
(d\star + \rho_{\mathrm{D}}(N))B^{\|} &= -2\tanh(t/2)B^{\|}\\
B^{\|}\cdot(d\star + \rho_{\mathrm{D}}(N))B^{\|} &= -2\frac{\tanh(t/2)}{\cosh^8(t/2)}\\
B^{\|}\cdot(d\star + \rho_{\mathrm{D}}(N))B^{\|}(d\mathrm{vol}_{\mathbb{S}^3(r)}) &= -2\frac{\tanh(t/2)}{\cosh^8(t/2)}\sinh^3(t)(d\mathrm{vol}_{\mathbb{S}^3})\\
\varepsilon_{\mathrm{top}} = \frac{1}{2}\int B^{\|}\cdot(d\star + \rho_{\mathrm{D}}(N))B^{\|}(d\mathrm{vol}(t)) &= -16\pi^2\frac{\tanh^4(t/2)}{\cosh^2(t/2)}.
\end{aligned}$$

Pointwise we have

$$\tfrac{1}{2}\Delta|B|^2 + |\nabla_A B|^2 = \langle B\cdot\nabla_A^*\nabla_A B\rangle.$$

By the Weitzenbock formula,

$$\langle B\cdot\nabla_A^*\nabla_A B\rangle = \langle B\cdot((d_A^*d_A + d_A d_A^*)B + \rho_{\mathrm{W}}(R)B - [F_A \,\mathbf{.}\, B])\rangle.$$

For a solution,

$$\tfrac{1}{8}\Delta|B|^2 + \tfrac{1}{4}|\nabla_A B|^2 + \tfrac{1}{32}|[B\,\mathbf{.}\,B]|^2 + \tfrac{1}{12}s|B|^2 + \tfrac{1}{2}\langle B\cdot\rho_{\mathrm{W}}(W^+)B\rangle = 0.$$

The radial Laplacian on $\mathbb{H}^4$ is

$$\Delta f(r) = -\left(f''(r) + \frac{3f'(r)}{\tanh(r)}\right).$$

We verify for our example that

$$\begin{aligned}
0 = \tfrac{1}{8}&\left((20 - 4\cosh(t))\mathrm{sech}^{10}(t/2)\right) + \tfrac{1}{4}(12\mathrm{sech}^8(t/2)\tanh^2(t/2)) +\\
&+ \tfrac{1}{32}(0) + \tfrac{1}{12}(-12)(2\mathrm{sech}^8(t/2)) + \tfrac{1}{2}0.
\end{aligned}$$

In the case $W^+ = 0$, we get the simple differential inequality:

$$\tfrac{1}{8}\Delta|B|^2 + \tfrac{1}{4}|\nabla|B||^2 \le -\tfrac{1}{12}s|B|^2 - \tfrac{1}{32}|[B\,\mathbf{.}\,B]|^2.$$

Note the resemblance to the harmonic oscillator $-f'' = -sf$ with $f = |B|^2$. This should effectively cut off high frequencies, leaving only wavelengths larger than $\approx \sqrt{-s}$.

$$\tfrac{1}{8}\Delta |B|^2 + \tfrac{1}{4} |\nabla |B||^2 \leq (\operatorname{dist}(\tilde{B}, Z)^2 - R) |B|^2 .$$

Chapter 7

The Kähler case

7.1 The Vafa-Witten equations on a Kähler manifold

An oriented Riemannian four-manifold X is a *Kähler* if it there exists $\omega \in \Omega^{2,+}(X;\mathbb{R})$ such that $\nabla\omega = 0$ and $|\omega| = 2$.

Given an orthonormal coframe $\{e^0, e^1, e^2, e^3\}$ for which $\omega = e^{01} + e^{23}$, we define

$$\begin{aligned} dz^1 &:= e^0 + ie^1, & dz^2 &:= e^2 + ie^3, \\ d\bar{z}^1 &:= e^0 - ie^1, & d\bar{z}^2 &:= e^2 - ie^3, \end{aligned}$$

so that

$$\omega = \tfrac{1}{2} i (dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2) = e^{01} + e^{23}.$$

Theorem 7.1.1. *When X is Kähler, the Vafa-Witten equations are*

$$\begin{aligned} \tfrac{1}{2}\omega^2 \left(i\Lambda F_A + \tfrac{1}{2}[\gamma, \gamma^*] \right) + \tfrac{1}{2}[\beta \wedge \beta^*] &= 0, \\ F_A^{2,0} - \tfrac{1}{2}[\gamma, \beta] &= 0, \\ \partial_A^* \beta - \partial_A \gamma^* &= 0, \end{aligned}$$

for $\gamma \in \Omega^0(X; \mathfrak{g}_P \otimes \mathbb{C})$ and $\beta \in \Omega^{2,0}(X; \mathfrak{g}_P \otimes \mathbb{C})$.

Note the $\mathrm{U}(1)$ symmetry given by $\beta \mapsto e^{i\theta}\beta$, $\gamma \mapsto e^{-i\theta}\gamma$.

Theorem 7.1.2. *If X is a* closed *Kähler manifold, these equations are equivalent to*

$$\begin{aligned} F_A &\in \mathcal{A}^{1,1}, & d_A\gamma &= 0, \\ \bar{\partial}_A \beta &= 0, & [\gamma, \gamma^*] &= 0, \\ \omega \wedge iF_A + \tfrac{1}{2}[\beta \wedge \beta^*] &= 0, & [\gamma, \beta + \beta^*] &= 0. \end{aligned}$$

In most interesting cases we have $\gamma = 0$ so that only the equations in the left column are relevant. For example, if A is an irreducible $\mathrm{SU}(2)$ connection, then $\ker d_A = 0$, so $\gamma = 0$.

Proof of Theorem 7.1.1 . We define β and γ so that if

$$B = B_1(e^{01} + e^{23}) + B_2(e^{02} + e^{31}) + B_3(e^{03} + e^{12}),$$

then

$$\begin{aligned}
\beta &:= \tfrac{1}{2}(B_2 - iB_3)dz^1 \wedge dz^2 \\
&= \tfrac{1}{2}(B_2 - iB_3)((e^{02} + e^{31}) + i(e^{03} + e^{12})) \\
\beta^* &:= -\overline{\beta} = -\tfrac{1}{2}(B_2 + iB_3)d\bar{z}^1 \wedge d\bar{z}^2, \\
&= -\tfrac{1}{2}(B_2 + iB_3)((e^{02} + e^{31}) - i(e^{03} + e^{12})) \\
\gamma &:= C - iB_1, \\
\gamma^* &:= -C - iB_1,
\end{aligned}$$

It follows that

$$B := B_1\omega + \beta - \beta^*.$$

Now we write out components of part of the Vafa-Witten equations

$$\begin{aligned}
-\tfrac{1}{4}\left[B \centerdot B\right] - \left[B, C\right] = &\left(\left[B_2, B_3\right] + \left[C, B_1\right]\right)\left(e^{01} + e^{23}\right) + \\
&+ \left(\left[B_3, B_1\right] + \left[C, B_2\right]\right)\left(e^{02} + e^{31}\right) + \\
&+ \left(\left[B_1, B_2\right] + \left[C, B_3\right]\right)\left(e^{03} + e^{12}\right),
\end{aligned}$$

and match them to expressions in β and γ:

$$\begin{aligned}
\tfrac{1}{2}i\left[\gamma, \gamma^*\right] &= \left[C, B_1\right], \\
\tfrac{1}{2}\left[\beta \centerdot \beta^*\right] &= \left[B_2, B_3\right]\left(e^{01} + e^{23}\right), \\
\tfrac{1}{2}i\left[\beta \wedge \beta^*\right] &= \left[B_2, B_3\right]e^{0123}, \\
\tfrac{1}{2}\omega \wedge \omega &= e^{0123}, \\
\left[\gamma, \beta\right] &= \tfrac{1}{2}\left(\left(\left[C, B_2\right] + \left[B_3, B_1\right]\right) - i\left(\left[B_1, B_2\right] + \left[C, B_3\right]\right)\right)\left((e^{02} + e^{31}) + i(e^{03} + e^{12})\right), \\
\left[\gamma, \beta\right]^* &= -\left[\gamma^*, \beta^*\right] \\
&= -\tfrac{1}{2}\left(\left(\left[C, B_2\right] + \left[B_3, B_1\right]\right) + i\left(\left[B_1, B_2\right] + \left[C, B_3\right]\right)\right)\left((e^{02} + e^{31}) - i(e^{03} + e^{12})\right), \\
\left[\gamma, \beta\right] - \left[\gamma, \beta\right]^* &= \left(\left[C, B_2\right] + \left[B_3, B_1\right]\right)\left(e^{02} + e^{31}\right) + \left(\left[B_1, B_2\right] + \left[C, B_3\right]\right)\left(e^{03} + e^{12}\right), \\
-\tfrac{1}{4}\left[B \centerdot B\right] - \left[B, C\right] &= \tfrac{1}{2}\left[\beta \centerdot \beta^*\right] + \tfrac{1}{2}i\left[\gamma, \gamma^*\right]\omega + \left[\gamma, \beta\right] - \left[\gamma, \beta\right]^*. \\
(d_A^* B + d_A C)^{1,0} &= \bar{\partial}_A^* L B_1 + \partial_A^* \beta + \partial C = \partial_A^* \beta + i\partial B_1 + \partial C = \partial^* \beta - \partial \gamma^*.
\end{aligned}$$

Thus the Vafa-Witten equations on a Kähler manifold are equivalent to

$$\begin{aligned}
\tfrac{1}{2}\omega^2\left(i\Lambda F_A + \tfrac{1}{2}\left[\gamma, \gamma^*\right]\right) + \tfrac{1}{2}\left[\beta \wedge \beta^*\right] &= 0, \\
F_A^{2,0} - \tfrac{1}{2}\left[\gamma, \beta\right] &= 0, \\
\partial_A^* \beta - \partial_A \gamma^* &= 0.
\end{aligned}$$

□

Proof of Theorem 7.1.2 . We use a series of integrations by parts, assuming that X is closed.

$$\begin{aligned}\|\partial_A^*\beta - \partial_A\gamma^*\|^2 &= \|\partial_A^*\beta\|^2 + \|\partial_A\gamma^*\|^2 - 2\int_X \langle \partial_A^*\beta \cdot \partial_A\gamma^*\rangle \\ &= \left\|\bar\partial_A\beta\right\|^2 + \left\|\bar\partial_A\gamma\right\|^2 - 2\int_X \left\langle \beta \cdot \left[F_A^{2,0}, \gamma^*\right]\right\rangle \\ &= \left\|\bar\partial_A\beta\right\|^2 + \left\|\bar\partial_A\gamma\right\|^2 + 2\int_X \left\langle F_A^{2,0} \cdot [\gamma, \beta]\right\rangle .\end{aligned}$$

$$2\left\|F_A^{2,0} - \tfrac{1}{2}[\gamma,\beta]\right\|^2 = 2\left\|F_A^{2,0}\right\|^2 + \tfrac{1}{2}\|[\gamma,\beta]\|^2 - 2\int_X \left\langle F_A^{2,0} \cdot [\gamma,\beta]\right\rangle .$$

$$\begin{aligned}&\left\|\tfrac{1}{2}\omega^2\left(i\Lambda F_A + \tfrac{1}{2}[\gamma,\gamma^*]\right) + \tfrac{1}{2}[\beta\wedge\beta^*]\right\|^2 \\ &= \|\omega \wedge iF_A + \tfrac{1}{2}[\beta\wedge\beta^*]\|^2 + \tfrac{1}{4}\|[\gamma,\gamma^*]\|^2 + \int_X \left(\langle i\Lambda F_A, [\gamma,\gamma^*]\rangle + \tfrac{1}{2}\langle [\gamma,\gamma^*], [\beta\wedge\beta^*]\rangle\right).\end{aligned}$$

$$\begin{aligned}\int_X \langle i\Lambda F_A \cdot [\gamma,\gamma^*]\rangle &= \int_X \langle i\Lambda [F_A,\gamma]\cdot\gamma\rangle \\ &= \int_X \left\langle i(\bar\partial_A\partial_A + \partial_A\bar\partial_A)\phi \cdot L\phi\right\rangle \\ &= \int_X \left\langle i\partial_A\phi \cdot \bar\partial_A^* L\phi\right\rangle + \int_X \left\langle i\bar\partial_A\phi \cdot \partial_A^* L\phi\right\rangle \\ &= \|\partial_A\gamma\|^2 - \left\|\bar\partial_A\gamma\right\|^2 .\end{aligned} \tag{7.1}$$

$$\begin{aligned}&\tfrac{1}{2}\langle [\gamma,\gamma^*], [\beta\wedge\beta^*]\rangle \\ &= -\tfrac{1}{2}\langle \gamma, [\gamma,[\beta\wedge\beta^*]]\rangle \\ &= -\tfrac{1}{2}\langle \gamma, [[\gamma,\beta]\wedge\beta^*]\rangle - \tfrac{1}{2}\langle \gamma, [\beta\wedge[\gamma,\beta^*]]\rangle \\ &= \tfrac{1}{2}\langle [\gamma,\beta]\wedge[\beta,\gamma]\rangle + \tfrac{1}{2}\langle [\gamma,\beta^*]\wedge[\gamma,\beta^*]\rangle \\ &= \tfrac{1}{2}\|[\gamma,\beta^*]\|^2 - \tfrac{1}{2}\|[\gamma,\beta]\| .\end{aligned}$$

Thus we get two identities when X is closed:

$$\begin{aligned}\|\partial_A^*\beta - \partial_A\gamma^*\|^2 + 2\left\|F_A^{2,0} - \tfrac{1}{2}[\gamma,\beta]\right\|^2 &= \left\|\bar\partial_A\gamma\right\|^2 + \tfrac{1}{2}\|[\gamma,\beta]\|^2 + \left\|\bar\partial_A\beta\right\|^2 + 2\left\|F_A^{2,0}\right\|^2, \\ \left\|\tfrac{1}{2}\omega^2\left(i\Lambda F_A + \tfrac{1}{2}[\gamma,\gamma^*]\right) + \tfrac{1}{2}[\beta\wedge\beta^*]\right\|^2 &= -\left\|\bar\partial_A\gamma\right\|^2 - \tfrac{1}{2}\|[\gamma,\beta]\| + \|\partial_A\gamma\|^2 + \\ &\quad + \|\omega\wedge iF_A + \tfrac{1}{2}[\beta\wedge\beta^*]\|^2 + \tfrac{1}{4}\|[\gamma,\gamma^*]\|^2 + \tfrac{1}{2}\|[\gamma,\beta^*]\|^2 .\end{aligned}$$

The left hand sides both vanish on solutions. The right hand side of the first equation is a sum of positive terms, and so they must vanish individually. The same is true for the sum of both equations. Thus all terms appearing here must vanish. □

7.2 Semistability

In this section, we prove an analogue of [Bra91, Theorem 2.1.6] for the Vafa-Witten equations.

We extend the Hermitian conjugate * to forms so that $(dz^1 \wedge dz^2)^* = d\bar{z}^1 \wedge d\bar{z}^2$. That way,

$$(dz^1 \wedge dz^2) \wedge (dz^1 \wedge dz^2)^* = dz^1 \wedge dz^2 \wedge d\bar{z}^1 \wedge d\bar{z}^2 = +4\, d\mathrm{vol} = +2\omega^2.$$

Similarly, any $\beta \in \Omega^{2,0}(\mathrm{Hom}(E,F))$ satisfies $\int \beta \wedge \beta^* \geq 0$. In contrast, if $N \in \Omega^{1,0}(\mathrm{Hom}(E,F))$, then $\int \omega \wedge N^* \wedge N \leq 0$.

Definition 7.2.1. Let E be a vector bundle over a Kähler four-manifold X. Then the *degree* of E is

$$\deg(E) := \langle c_1(E) \smile [\omega], [X] \rangle = \int_X \frac{i}{2\pi} \mathrm{Tr}(F_A) \wedge \omega,$$

for any connection A on E.

Definition 7.2.2. We define the *slope*

$$\mu(E) := \frac{\deg(E)}{\mathrm{rank}(E)}.$$

Observe that $\Omega^{2,0}(\mathrm{End}(E)) \cong \Omega^0(\mathrm{Hom}(E, E \otimes K))$. For any $\beta \in \Omega^{2,0}(\mathrm{End}(E))$, we say that a subbundle $E' \subset E$ is *β-invariant* if

$$\beta(E') \subset E' \otimes K.$$

Definition 7.2.3. A holomorphic vector bundle E is *β-semistable* if all β-invariant holomorphic subbundles $E' \subset E$ satisfy $\mu(E') \leq \mu(E)$.

Theorem 7.2.4. *Let A be a holomorphic connection on a Hermitian vector bundle E of rank R. Let F_A^0 denote the traceless part of F_A. If $\beta \in \Omega^{2,0}(\mathrm{End}(E))$ is a solution to*

$$\omega \wedge iF_A^0 + \tfrac{1}{2}[\beta \wedge \beta^*] = 0, \tag{7.2}$$

then E is β-semistable. Furthermore, if E' is a β-invariant holomorphic subbundle such that $\mu(E') \leq \mu(E)$, then the orthogonal complement $E^\perp$ is holomorphic, and the restrictions of β to E' and $E^\perp$ both satisfy (7.2).

Proof. We deduce the result by studying the restriction of this equation to a holomorphic subbundle E'.

Let E' be a holomorphic subbundle of E, and let $E^\perp$ be its orthogonal complement. The connection decomposes as

$$A = \begin{pmatrix} A' & -N^* \\ N & A^\perp \end{pmatrix}$$

with $N \in \Omega^{1,0}(E^\perp \otimes E'^*)$. Curvature decomposes as

$$F_A = \begin{pmatrix} F_A' - N^* \wedge N & \bullet \\ \bullet & F_A^\perp - N \wedge N^* \end{pmatrix}.$$

Note that $i\omega \wedge \operatorname{Tr}(-N^* \wedge N) \geq 0$.

If $\beta = \begin{pmatrix} \beta' & \beta_{12} \\ \beta_{21} & \beta^\perp \end{pmatrix}$, then the restriction of $[\beta \wedge \beta^*]$ to E' is

$$[\beta \wedge \beta^*]' = [\beta' \wedge \beta'^*] + \beta_{12} \wedge \beta_{12}^* - \beta_{21}^* \wedge \beta_{21}.$$

Invariance of E' by β means that $\beta_{21} = 0$. Since the trace vanishes on commutators, it follows that if E' is β-invariant, then $\operatorname{Tr}[\beta \wedge \beta^*]' = \operatorname{Tr}(\beta_{12} \wedge \beta_{12}^*) \geq 0$.

The restriction of (7.2) to a β-invariant subbundle E' is thus

$$\omega \wedge i\,(F'_A - N^* \wedge N - \operatorname{Tr}(F_A)/R) + [\beta' \wedge \beta'^*] + \beta_{12} \wedge \beta_{12}^* = 0.$$

Integrating the trace, we get

$$2\pi(\deg(E') - (r/R)\deg(E)) + \int_X (i\omega \wedge \operatorname{Tr}(-N^* \wedge N) + \operatorname{Tr}(\beta_{12} \wedge \beta_{12}^*)) = 0.$$

The integrand is nonnegative, so we get

$$\mu(E') \leq \mu(E),$$

with equality if and only if the integral is zero. The integrand is zero only when both N and β_{12} both vanish identically. The vanishing of N is equivalent to holomorphicity of $E^\perp$, and if β_{12} vanishes, then β splits as $\beta = \beta' \oplus \beta^\perp$. $\square$

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Chapter 8

Dimensional reduction

In this chapter, we compute the dimensional reduction of the Vafa-Witten equations to dimensions three and two, following a procedure similar to that of Hitchin in [Hit87]. We discover that the reduction (8.6) to a closed Riemann surface is essentially Hitchin's equations for Higgs pairs (A, Φ).

8.1 Hitchin's equations and the reduction of Yang-Mills

Though motivated by dimensional reduction, Hitchin's equations for Higgs pairs are distinct from the dimensional reduction of the Yang-Mills equation. The dimensional reduction of Yang-Mills is given by [Hit87] (1.2),

$$\begin{aligned} F_A - \tfrac{1}{2} i \left[\phi, \phi^*\right] d\mathrm{vol}_\Sigma &= 0, \\ \bar{\partial}_A \phi &= 0, \end{aligned} \tag{8.1}$$

for a principal bundle $P \to \Sigma$, a connection $A \in \mathcal{A}_P$ and $\phi \in \Omega^0(\Sigma; \mathfrak{g}_P \otimes \mathbb{C})$.

In contrast, Hitchin's equations are [Hit87] (1.3),

$$\begin{aligned} F_A + \left[\Phi, \Phi^*\right] &= 0, \\ \bar{\partial}_A \Phi &= 0, \end{aligned} \tag{8.2}$$

for $\Phi \in \Omega^{1,0}(\Sigma; \mathfrak{g}_P \otimes \mathbb{C})$.

The equations (8.1) are not that interesting, due to the the following fact:

Theorem 8.1.1. *Any solution (A, ϕ) to* (8.1) *on a closed surface Σ also satisfies $F_A = 0$ and $d_A \phi = 0$.*

Proof. Any solution of (8.1) must satisfy

$$\begin{aligned} 0 &= \left\| \bar{\partial}_A \phi \right\|^2 + \left\| F_A - \tfrac{1}{2} i \left[\phi, \phi^*\right] d\mathrm{vol}_\Sigma \right\|^2 \\ &= \left\| \bar{\partial}_A \phi \right\|^2 + \left\| F_A \right\|^2 + \tfrac{1}{4} \left\| \left[\phi, \phi^*\right] \right\|^2 + \int_\Sigma \left\langle \left[\phi, \phi^*\right], iF_A \right\rangle . \end{aligned}$$

Here we extend $\langle \bullet, \bullet \rangle$ to be complex linear in the first component and conjugate linear in the second component.

Now we work on the last term

$$\begin{aligned}\int_\Sigma \langle [\phi, \phi^*], iF_A \rangle &= \int_\Sigma \langle \phi, [iF_A, \phi] \rangle \\ &= \int_\Sigma \left\langle \phi, i(\bar{\partial}_A \partial_A + \partial_A \bar{\partial}_A)\phi \right\rangle \\ &= \int_\Sigma \left(\partial \langle \phi, i\partial_A \phi \rangle - \langle \partial_A \phi \wedge i\partial_A \phi \rangle + \bar{\partial} \left\langle \phi, i\bar{\partial}_A \phi \right\rangle - \left\langle \bar{\partial}_A \phi, i\bar{\partial}_A \phi \right\rangle \right) \\ &= \|\partial_A \phi\|^2 - \left\|\bar{\partial}_A \phi\right\|^2,\end{aligned}$$

where to get the last line we assumed that Σ has no boundary, and used the identities

$$\begin{aligned}-dz \wedge \overline{i\, dz} &= (dz \cdot dz)\, d\mathrm{vol}_\Sigma, \\ -d\bar{z} \wedge \overline{i\, d\bar{z}} &= -(d\bar{z} \cdot d\bar{z})\, d\mathrm{vol}_\Sigma.\end{aligned}$$

Thus

$$0 = \|F_A\|^2 + \tfrac{1}{4} \|[\phi, \phi^*]\|^2 + \|\partial_A \phi\|^2,$$

so F_A and $\partial_A \phi$ must also vanish, and also $d_A \phi = (\partial_A + \bar{\partial}_A)\phi = 0$. □

The reduction of the Vafa-Witten equations is like a combination of (8.1) and (8.2). The connection A over X splits into both $\phi \in \Omega^0(\Sigma; \mathfrak{g}_P \otimes \mathbb{C})$ and a connection A^Σ over Σ. Two of three components of B combine into a Higgs field $\Phi \in \Omega^{1,0}(\Sigma; \mathfrak{g}_P \otimes \mathbb{C})$, while the remaining component of B combines with C to form $\gamma \in \Omega^0(\Sigma; \mathfrak{g}_P \otimes \mathbb{C})$. The precise combinations are described in (8.8).

As was the case in Theorem 8.1.1, the raw reduced Vafa-Witten equations (8.7) simplify considerably (8.6) after integration by parts, and the only interesting equations which survive are Hitchin's equations (8.2).

8.2 Reduction to three dimensions

Let Y be an oriented Riemannian three-manifold with a principal bundle $P \to Y$. Let X denote $\mathbb{R} \times Y$ with the product metric and $\mathbb{R}$-coordinate x^0. Over X, the Levi-Civita derivative ∇_0 reduces to the Lie derivative $\mathcal{L}_0$. On the pullback of P to X, the Lie derivative $\mathcal{L}_0$ extends to a partial connection, which we also denote by $\mathcal{L}_0$. We think of $\mathcal{L}_0$ as the "time derivative." We identify objects over Y with their $\mathcal{L}_0$-invariant pullbacks to X.

Given a connection A^Y on P and a section $A_0 \in \Omega^0(Y; \mathfrak{g}_P)$, we get a connection A on the pullback of P over X given by

$$\nabla_A = dx^0 \otimes (\mathcal{L}_0 + A_0) + \nabla_{A^Y}, \tag{8.3}$$

and satisfies $[\mathcal{L}_0, \nabla_A] = 0$. Conversely, any connection satisfying $[\mathcal{L}_0, \nabla_A] = 0$ has this form.

Now consider over $X = \mathbb{R} \times Y$ the Vafa-Witten equations

$$\begin{aligned} 2F_A^+ + \tfrac{1}{4}[B \mathbin{.} B] + [B, C] &= 0, \\ d_A^* B + d_A C &= 0, \end{aligned} \tag{8.4}$$

together with the dimensional reduction equations

$$[\mathcal{L}_0, \nabla_A] = 0, \quad \mathcal{L}_0 B = 0, \quad \mathcal{L}_0 C = 0. \tag{8.5}$$

Theorem 8.2.1. *Any solution to* (8.4) *and* (8.5) *pulls back from a solution to the three-dimensional Vafa-Witten equations is*

$$\begin{aligned} F_{A^Y} - \star d_{A^Y} A_0 - \tfrac{1}{2}\left[\tilde{B} \wedge \tilde{B}\right] + \left[\star \tilde{B}, C\right] &= 0, \\ d_{A^Y} C + \star d_{A^Y} \tilde{B} - \left[A_0, \tilde{B}\right] &= 0, \\ d_{A^Y}^* \tilde{B} - [A_0, C] &= 0, \end{aligned}$$

where A^Y is a connection over Y, $A_0, C \in \Omega^0(Y; \mathfrak{g}_P)$, and $\tilde{B} \in \Omega^1(Y; \mathfrak{g}_P)$.

This theorem is a direct consequence of

Lemma 8.2.2. *Let A be the connection defined by* (8.3)*, and $\tilde{B} := e^0 \mathbin{.} B$. If A_0, $\tilde{B}$, and C are invariant under $\mathcal{L}_0$, then these fields are pulled back from Y and satisfy*

$$\begin{aligned} 2F_A^+ + \tfrac{1}{4}[B \mathbin{.} B] + [B, C] &= \left(1 + \star^X\right)\left(F_{A^Y} - \star^Y d_{A^Y} A_0 - \tfrac{1}{2}\left[\tilde{B} \wedge \tilde{B}\right] + \left[\star^Y \tilde{B}, C\right]\right), \\ d_A^* B + d_A C &= \left(d_{A^Y} C + \star^Y d_{A^Y} \tilde{B} - \left[A_0, \tilde{B}\right]\right) - e^0 \wedge \left(d_{A^Y}^* \tilde{B} - [A_0, C]\right). \end{aligned}$$

Proof. From the definition of the reduced ∇_A on Y, we compute

$$F_A = d_A d_A = F_{A^Y} - e^0 \wedge d_{A^Y} A_0.$$

It follows that

$$\begin{aligned} 2F_A^+ &= \left(1 + \star^X\right)\left(F_{A^Y} - dx^0 \wedge d_{A^Y} A_0\right) \\ &= \left(1 + \star^X\right)\left(F_{A^Y} - \star^X\left(dx^0 \wedge d_{A^Y} A_0\right)\right) \\ &= \left(1 + \star^X\right)\left(F_{A^Y} - \star^Y d_{A^Y} A_0\right). \end{aligned}$$

For

$$B = B_1(e^{01} + e^{23}) + B_2(e^{02} + e^{31}) + B_3(e^{03} + e^{12}),$$

we get

$$\tilde{B} = e^0 \mathbin{.} B = B_1 e^1 + B_2 e^2 + B_3 e^3.$$

Then

$$\star^Y \tilde{B} = B_1 e^{23} + B_2 e^{31} + B_3 e^{12},$$

and

$$B = \left(1 + \star^X\right) \star^Y \tilde{B}.$$

Note that

$$\tfrac{1}{2}\left[\tilde{B}\wedge\tilde{B}\right] = [B_2, B_3]\, e^{23} + [B_3, B_1]\, e^{31} + [B_1, B_2]\, e^{12}.$$

Recall that

$$-\tfrac{1}{4}\left[B \mathbin{\scriptstyle\blacksquare} B\right] = [B_2, B_3]\,(e^{01} + e^{23}) + [B_3, B_1]\,(e^{02} + e^{31}) + [B_1, B_2]\,(e^{03} + e^{12}).$$

Thus

$$\tfrac{1}{4}\left[B \mathbin{\scriptstyle\blacksquare} B\right] = -\tfrac{1}{2}(1 + \star^X)\left[\tilde{B}\wedge\tilde{B}\right].$$

It follows that

$$2F_A^+ + \tfrac{1}{4}\left[B \mathbin{\scriptstyle\blacksquare} B\right] + [B, C] = \left(1 + \star^X\right)\left(F_{A^Y} - \star^Y d_{A^Y} A_0 - \tfrac{1}{2}\left[\tilde{B}\wedge\tilde{B}\right] + \left[\star^Y \tilde{B}, C\right]\right).$$

Next we reduce the second equation.

$$\begin{aligned} d_A^* B + d_A C &= -\left[A_0, e^0 \mathbin{\scriptstyle\blacksquare} B\right] + d_{A^Y}^* B + e^0 \wedge [A_0, C] + d_{A^Y} C \\ &= \left(d_{A^Y} C + \star^Y d_{A^Y} \tilde{B} - \left[A_0, \tilde{B}\right]\right) - e^0 \wedge \left(d_{A^Y}^* \tilde{B} - [A_0, C]\right). \end{aligned}$$

□

8.3 Reduction to two dimensions

Continuing further, we reduce to a Riemann surface Σ.

Theorem 8.3.1. *On a closed Riemann surface Σ, the two-dimensional reduction of the Vafa-Witten equations is equivalent to*

$$\begin{aligned} &\bar{\partial}_{A^\Sigma}\Phi = 0, && F_{A^\Sigma} + [\Phi\wedge\Phi^*] = 0, \\ &\partial_{A^\Sigma}\phi = 0, \quad [\phi, \phi^*] = 0, && [\Phi, \phi^*] = 0, \\ &\partial_{A^\Sigma}\gamma = 0, \quad [\gamma, \gamma^*] = 0, && [\Phi, \gamma^*] = 0, \\ &\qquad\qquad [\phi, \gamma^*] = 0, && [\gamma, \phi] = 0, \\ & && \bar{\partial}_{A^\Sigma}\phi - [\Phi^*, \gamma^*] = 0, \\ & && \bar{\partial}_{A^\Sigma}\gamma - [\phi^*, \Phi^*] = 0, \end{aligned} \tag{8.6}$$

where $\Phi \in \Omega^{1,0}(\Sigma; \mathfrak{g}_P \otimes \mathbb{C})$, and $\phi, \gamma \in \Omega^0(\Sigma; \mathfrak{g}_P \otimes \mathbb{C})$.

Thus a solution $(A^\Sigma, \Phi, \phi, \gamma)$ corresponds to a Higgs pair (A^Σ, Φ) plus some extra anti-holomorphic fields ϕ and γ satisfying various commutation relations.

We break the proof into three lemmas. We compute the reduced equations in Lemma 8.3.2. Then we rephrase the equations in the language of Kähler geometry in Lemma 8.3.3. To get these equations into the desired form, we assume that Σ is closed and integrate by parts in Lemma 8.3.5.

Lemma 8.3.2. *Let A^Y be a connection over $\mathbb{R}\times\Sigma$ with $\mathbb{R}$-coordinate x^1 such that $\nabla_{A^Y} = dx^1\otimes(\mathcal{L}_1 + A_1) + \nabla_{A^\Sigma}$, and $\tilde{\tilde{B}} := \tilde{B} - B_1 e^1$. If A_0, A_1, C, B_1, and $\tilde{\tilde{B}}$ are invariant under $\mathcal{L}_0$, then these fields are pulled back from Σ and satisfy*

$$\begin{aligned}&\left(1+\star^X\right)\Big(F_{A^\Sigma} + \star^\Sigma\left([A_0,A_1]+[B_1,C]\right) - \tfrac{1}{2}\left[\tilde{\tilde{B}}\wedge\tilde{\tilde{B}}\right] + \\ &\quad + e^1\wedge\left(\star^\Sigma d_{A^\Sigma}A_0 - d_{A^\Sigma}A_1 - \left[B_1,\tilde{\tilde{B}}\right] - \left[\star^\Sigma\tilde{\tilde{B}},C\right]\right)\Big) = 2F_A^+ + \tfrac{1}{4}\left[B\,.\,B\right] + [B,C],\end{aligned}$$

$$\begin{aligned}&\left(d_{A^\Sigma}C - \star^\Sigma d_{A^\Sigma}B_1 - \left[A_0,\tilde{\tilde{B}}\right] + \left[A_1,\star^\Sigma\tilde{\tilde{B}}\right]\right) + \\ &\quad + e^0\wedge\left(-d^*_{A^\Sigma}\tilde{\tilde{B}} + [A_1,B_1] + [A_0,C]\right) + \\ &\quad + e^1\wedge\left(\star^\Sigma d_{A^\Sigma}\tilde{\tilde{B}} - [A_0,B_1] + [A_1,C]\right) = d_A^*B + d_AC.\end{aligned}$$

Lemma 8.3.3. *Define*

$$\begin{aligned}\phi &:= A_0 + iA_1,\\ \Phi &:= \tfrac{1}{2}(1+i\star^\Sigma)\tilde{\tilde{B}},\\ \gamma &:= C - iB_1.\end{aligned}$$

Then

$$\begin{aligned}&2F_A^+ + \tfrac{1}{4}\left[B\,.\,B\right] + [B,C]\\ &\quad = \left(1+\star^X\right)\left(F_{A^\Sigma} - \tfrac{1}{2}i\,\star^\Sigma\left([\phi,\phi^*]+[\gamma,\gamma^*]\right) + [\Phi\wedge\Phi^*] - 2e^1\wedge\operatorname{Im}\left(\bar\partial_{A^\Sigma}\phi - [\Phi^*,\gamma^*]\right)\right),\end{aligned}$$

$$d_A^*B + d_AC = 2\operatorname{Re}\left(\bar\partial_{A^\Sigma}\gamma - [\phi^*,\Phi^*]\right) + \operatorname{Im}((e^0+ie^1)\wedge\star^\Sigma(2\bar\partial_{A^\Sigma}\Phi + i\,\star^\Sigma[\gamma^*,\phi^*])).$$

Definition 8.3.4. To be consistent with the language of Kähler geometry on Σ, let $\omega := d\mathrm{vol}_\Sigma$ denote the Kähler form, $L := \omega\wedge$, and $\Lambda := L^*$ which is equivalent to $\Lambda : \Omega^2(\Sigma)\to\Omega^0(\Sigma)$ by $\Lambda := \star^\Sigma$.

Lemma 8.3.5. *On a closed Riemann surface Σ, the equations*

$$\begin{aligned}i\Lambda\left(F_{A^\Sigma} + [\Phi\wedge\Phi^*]\right) + \tfrac{1}{2}\left([\phi,\phi^*]+[\gamma,\gamma^*]\right) &= 0,\\ i\Lambda\bar\partial_{A^\Sigma}\Phi - \tfrac{1}{2}[\gamma^*,\phi^*] &= 0,\\ \bar\partial_{A^\Sigma}\phi - [\Phi^*,\gamma^*] &= 0,\\ \bar\partial_{A^\Sigma}\gamma - [\phi^*,\Phi^*] &= 0,\end{aligned} \tag{8.7}$$

are equivalent to

$$\begin{gathered}\bar\partial_{A^\Sigma}\Phi = 0,\qquad F_{A^\Sigma} + [\Phi\wedge\Phi^*] = 0,\\ \partial_{A^\Sigma}\phi = 0,\quad [\phi,\phi^*] = 0,\quad [\Phi,\phi^*] = 0,\\ \partial_{A^\Sigma}\gamma = 0,\quad [\gamma,\gamma^*] = 0,\quad [\Phi,\gamma^*] = 0,\\ [\phi,\gamma^*] = 0.\end{gathered}$$

Proof of Lemma 8.3.2. We compute

$$
\begin{aligned}
F_{A^Y} &= F_{A^\Sigma} - e^1 \wedge d_{A^\Sigma} A_1, \\
-\star^Y d_{A^Y} A_0 &= -\star^Y \left(e^1 \wedge [A_1, A_0] + d_{A^\Sigma} A_0\right) \\
&= \star^\Sigma [A_0, A_1] + e^1 \wedge \star^\Sigma d_{A^\Sigma} A_0, \\
-\tfrac{1}{2}\left[\tilde{B} \wedge \tilde{B}\right] &= -\tfrac{1}{2}\left[\tilde{\tilde{B}} \wedge \tilde{\tilde{B}}\right] - e^1 \wedge \left[B_1, \tilde{\tilde{B}}\right], \\
\left[\star^Y \tilde{B}, C\right] &= \star^\Sigma [B_1, C] - e^1 \wedge \left[\star^\Sigma \tilde{\tilde{B}}, C\right], \\
d_{A^Y} C &= e^1 \wedge [A_1, C] + d_{A^\Sigma} C, \\
\star^Y d_{A^Y} \tilde{B} &= \star^Y \left(-e^1 \wedge d_{A^\Sigma} B_1 + e^1 \wedge \left[A_1, \tilde{\tilde{B}}\right] + d_{A^\Sigma} \tilde{\tilde{B}}\right) \\
&= -\star^\Sigma d_{A^\Sigma} B_1 + \left[A_1, \star^\Sigma \tilde{\tilde{B}}\right] - e^1 \wedge \star^\Sigma d_{A^\Sigma} \tilde{\tilde{B}}, \\
-\left[A_0, \tilde{B}\right] &= -e^1 \wedge [A_0, B_1] - \left[A_0, \tilde{\tilde{B}}\right], \\
d^*_{A^Y} \tilde{B} &= d^*_{A^Y}\left(B_1 e^1 + \tilde{\tilde{B}}\right) = -[A_1, B_1] + d^*_{A^\Sigma} \tilde{\tilde{B}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&F_{A^Y} - \star^Y d_{A^Y} A_0 - \tfrac{1}{2}\left[\tilde{B} \wedge \tilde{B}\right] + \left[\star^Y \tilde{B}, C\right] \\
&= F_{A^\Sigma} + \star^\Sigma \left([A_0, A_1] + [B_1, C]\right) - \tfrac{1}{2}\left[\tilde{\tilde{B}} \wedge \tilde{\tilde{B}}\right] + \\
&\quad + e^1 \wedge \left(\star^\Sigma d_{A^\Sigma} A_0 - d_{A^\Sigma} A_1 - \left[B_1, \tilde{\tilde{B}}\right] - \left[\star^\Sigma \tilde{\tilde{B}}, C\right]\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&d_{A^Y} C + \star^Y d_{A^Y} \tilde{B} - \left[A_0, \tilde{B}\right] \\
&= \left(d_{A^\Sigma} C - \star^\Sigma d_{A^\Sigma} B_1 - \left[A_0, \tilde{\tilde{B}}\right] + \left[A_1, \star^\Sigma \tilde{\tilde{B}}\right]\right) + \\
&\quad + e^1 \wedge \left(\star^\Sigma d_{A^\Sigma} \tilde{\tilde{B}} - [A_0, B_1] + [A_1, C]\right),
\end{aligned}
$$

and finally

$$
d^*_{A^Y} \tilde{B} - [A_0, C] = d^*_{A^\Sigma} \tilde{\tilde{B}} - [A_1, B_1] - [A_0, C].
$$

The original equations become

$$
\begin{aligned}
\left(1 + \star^X\right)\Big(F_{A^\Sigma} &+ \star^\Sigma \left([A_0, A_1] + [B_1, C]\right) - \tfrac{1}{2}\left[\tilde{\tilde{B}} \wedge \tilde{\tilde{B}}\right] + \\
&+ e^1 \wedge \left(\star^\Sigma d_{A^\Sigma} A_0 - d_{A^\Sigma} A_1 - \left[B_1, \tilde{\tilde{B}}\right] - \left[\star^\Sigma \tilde{\tilde{B}}, C\right]\right)\Big) = 0, \\
\Big(d_{A^\Sigma} C &- \star^\Sigma d_{A^\Sigma} B_1 - \left[A_0, \tilde{\tilde{B}}\right] + \left[A_1, \star^\Sigma \tilde{\tilde{B}}\right]\Big) + \\
&+ e^0 \wedge \left(-d^*_{A^\Sigma} \tilde{\tilde{B}} + [A_1, B_1] + [A_0, C]\right) + \\
&+ e^1 \wedge \left(\star^\Sigma d_{A^\Sigma} \tilde{\tilde{B}} - [A_0, B_1] + [A_1, C]\right) = 0.
\end{aligned}
$$

$\square$

Proof of Lemma 8.3.3. We define

$$\begin{aligned} \phi &:= A_0 + iA_1, \\ \Phi &:= \tfrac{1}{2}(1 + i\star^\Sigma)\tilde{\tilde{B}}, \\ \gamma &:= C - iB_1. \end{aligned} \tag{8.8}$$

Let * denote the Hermitian conjugate, and note that

$$\begin{aligned} dz &:= e^2 + ie^3, \\ \Phi &= \tfrac{1}{2}(B_2 - iB_3)dz, \\ \tilde{\tilde{B}} &= \mathrm{Re}(\Phi), \\ \bar{\partial}_{A^\Sigma} &:= \tfrac{1}{2}d\bar{z}(\nabla_{A^\Sigma,2} + i\nabla_{A^\Sigma,3}), \\ \bar{\partial}_{A^\Sigma} &= \begin{cases} \tfrac{1}{2}\left(d_{A^\Sigma} - i \star^\Sigma d_{A^\Sigma}\right) & \text{on } \Omega^{0,0}(\Sigma;\mathfrak{g}_P), \\ d_{A^\Sigma} & \text{on } \Omega^{1,0}(\Sigma;\mathfrak{g}_P), \end{cases} \\ \phi^* &= -(A_0 - iA_1) \\ \Phi^* &= -\tfrac{1}{2}(B_2 + iB_3)d\bar{z} = -\tfrac{1}{2}(1 - \star^\Sigma i)\tilde{\tilde{B}}, \\ \gamma^* &= -(C + iB_1). \end{aligned}$$

Next we compute

$$\begin{aligned} 2(\bar{\partial}_{A^\Sigma}\phi - [\Phi^*, \gamma^*]) &= \left(d_{A^\Sigma} - i \star^\Sigma d_{A^\Sigma}\right)(A_0 + iA_1) - \left[(1 - \star^\Sigma i)\tilde{\tilde{B}}, C + iB_1\right] \\ &= \left(d_{A^\Sigma}A_0 + \star^\Sigma d_{A^\Sigma}A_1 - \left[\tilde{\tilde{B}}, C\right] - \left[\star^\Sigma\tilde{\tilde{B}}, B_1\right]\right) \\ &\quad + i\left(d_{A^\Sigma}A_1 - \star^\Sigma d_{A^\Sigma}A_0 - \left[\tilde{\tilde{B}}, B_1\right] + \left[\star^\Sigma\tilde{\tilde{B}}, C\right]\right) \\ &= -i(1 - i\star^\Sigma)\left(\star^\Sigma d_{A^\Sigma}A_0 - d_{A^\Sigma}A_1 - \left[B_1, \tilde{\tilde{B}}\right] - \left[\star^\Sigma\tilde{\tilde{B}}, C\right]\right). \end{aligned}$$

Also,

$$\begin{aligned} 2\bar{\partial}_{A^\Sigma}\Phi + i \star^\Sigma [\gamma^*, \phi^*] &= d_{A^\Sigma}(1 + i\star^\Sigma)\tilde{\tilde{B}} + i \star^\Sigma [C + iB_1, A_0 - iA_1] \\ &= d_{A^\Sigma}\tilde{\tilde{B}} - \star^\Sigma([B_1, A_0] - [C, A_1]) + i \star^\Sigma\left(d^*_{A^\Sigma}\tilde{\tilde{B}} + [C, A_0] + [B_1, A_1]\right). \end{aligned}$$

In particular,

$$\begin{aligned} \mathrm{Im}((e^0 + ie^1) \wedge \star^\Sigma(2\bar{\partial}_{A^\Sigma}\Phi + i \star^\Sigma [\gamma^*, \phi^*])) &= e^0 \wedge \left(-d^*_{A^\Sigma}\tilde{\tilde{B}} + [A_1, B_1] + [A_0, C]\right) + \\ &\quad + e^1 \wedge \left(\star^\Sigma d_{A^\Sigma}\tilde{\tilde{B}} - [A_0, B_1] + [A_1, C]\right). \end{aligned}$$

Also,

$$
\begin{aligned}
2\left(\bar{\partial}_{A^\Sigma}\gamma - [\phi^*, \Phi^*]\right) &= \left(d_{A^\Sigma} - i \star^\Sigma d_{A^\Sigma}\right)(C - iB_1) - \left[A_0 - iA_1, (1 - i\star^\Sigma)\tilde{\tilde{B}}\right] \\
&= d_{A^\Sigma} C - \star^\Sigma d_{A^\Sigma} B_1 - \left[A_0, \tilde{\tilde{B}}\right] + \left[A_1, \star^\Sigma \tilde{\tilde{B}}\right] + \\
&\quad + i\left(-d_{A^\Sigma} B_1 - \star^\Sigma d_{A^\Sigma} C + \left[A_0, \star^\Sigma \tilde{\tilde{B}}\right] + \left[A_1, \tilde{\tilde{B}}\right]\right) \\
&= (1 - i\star^\Sigma)\left(d_{A^\Sigma} C - \star^\Sigma d_{A^\Sigma} B_1 - \left[A_0, \tilde{\tilde{B}}\right] + \left[A_1, \star^\Sigma \tilde{\tilde{B}}\right]\right).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&2F_{A^\Sigma} - i \star^\Sigma \left([\phi, \phi^*] + [\gamma, \gamma^*]\right) + 2\left[\Phi \wedge \Phi^*\right] \\
&= 2F_{A^\Sigma} + 2 \star^\Sigma \left([A_0, A_1] + [B_1, C]\right) - \left[\tilde{\tilde{B}} \wedge \tilde{\tilde{B}}\right].
\end{aligned}
$$

The original equations become

$$
\begin{aligned}
&2F_A^+ + \tfrac{1}{4}[B \mathbin{.} B] + [B, C] \\
&\quad = \left(1 + \star^X\right)\left(F_{A^\Sigma} - \tfrac{1}{2} i \star^\Sigma \left([\phi, \phi^*] + [\gamma, \gamma^*]\right) + [\Phi \wedge \Phi^*] - 2e^1 \wedge \operatorname{Im}\left(\bar{\partial}_{A^\Sigma}\phi - [\Phi^*, \gamma^*]\right)\right), \\
&d_A^* B + d_A C = 2\operatorname{Re}\left(\bar{\partial}_{A^\Sigma}\gamma - [\phi^*, \Phi^*]\right) + \operatorname{Im}((e^0 + ie^1) \wedge \star^\Sigma(2\bar{\partial}_{A^\Sigma}\Phi + i \star^\Sigma [\gamma^*, \phi^*])).
\end{aligned}
$$

$\square$

A solution to the original equations is thus equivalent to

$$
\begin{aligned}
i\Lambda\left(F_{A^\Sigma} + [\Phi \wedge \Phi^*]\right) + \tfrac{1}{2}\left([\phi, \phi^*] + [\gamma, \gamma^*]\right) &= 0, \\
i\Lambda\bar{\partial}_{A^\Sigma}\Phi - \tfrac{1}{2}[\gamma^*, \phi^*] &= 0, \\
\bar{\partial}_{A^\Sigma}\phi - [\Phi^*, \gamma^*] &= 0, \\
\bar{\partial}_{A^\Sigma}\gamma - [\phi^*, \Phi^*] &= 0.
\end{aligned}
$$

In the special case when Σ is flat, these equations take a particularly symmetric form. Writing $\tilde{\Phi} := B_2 - iB_3 = 2\Phi/dz$, we get

$$
\begin{aligned}
i\Lambda F_{A^\Sigma} + \tfrac{1}{2}\left([\phi, \phi^*] + \left[\tilde{\Phi} \wedge \tilde{\Phi}^*\right] + [\gamma, \gamma^*]\right) &= 0, \\
\bar{\partial}_{A^\Sigma}\phi - \left[\tilde{\Phi}^*, \gamma^*\right] &= 0, \\
\bar{\partial}_{A^\Sigma}\tilde{\Phi} - [\gamma^*, \phi^*] &= 0, \\
\bar{\partial}_{A^\Sigma}\gamma - \left[\phi^*, \tilde{\Phi}^*\right] &= 0.
\end{aligned}
$$

Proof of Lemma 8.3.5. The norm of the first equation is

$$\begin{aligned}
&\|F_{A^\Sigma} + [\Phi \wedge \Phi^*]\|^2 + \tfrac{1}{4}\|[\phi, \phi^*]\|^2 + \tfrac{1}{4}\|[\gamma, \gamma^*]\|^2 + \\
&+ \int_\Sigma \langle i\Lambda [\Phi \wedge \Phi^*], [\phi, \phi^*]\rangle + \int_\Sigma \langle i\Lambda [\Phi \wedge \Phi^*], [\gamma, \gamma^*]\rangle + \tfrac{1}{2}\int_\Sigma \langle [\phi, \phi^*], [\gamma, \gamma^*]\rangle + \\
&+ \int_\Sigma \langle i\Lambda F_{A^\Sigma}, [\phi, \phi^*]\rangle + \int_\Sigma \langle i\Lambda F_{A^\Sigma}, [\gamma, \gamma^*]\rangle .
\end{aligned}$$

Twice the norm of the second equation is

$$2\left\|\bar\partial_{A^\Sigma}\Phi\right\|^2 + \tfrac{1}{2}\|[\gamma^*, \phi^*]\|^2 - 2\int_\Sigma \left\langle i\Lambda\bar\partial_{A^\Sigma}\Phi, [\gamma^*, \phi^*]\right\rangle .$$

The norms of the third and fourth equations are

$$\begin{aligned}
&\left\|\bar\partial_{A^\Sigma}\phi\right\|^2 + \|[\Phi^*, \gamma^*]\|^2 - 2\int_\Sigma \left\langle \bar\partial_{A^\Sigma}\phi, [\Phi^*, \gamma^*]\right\rangle , \\
&\left\|\bar\partial_{A^\Sigma}\gamma\right\|^2 + \|[\phi^*, \Phi^*]\|^2 - 2\int_\Sigma \left\langle \bar\partial_{A^\Sigma}\gamma, [\phi^*, \Phi^*]\right\rangle .
\end{aligned}$$

Adding together all these norms, we get

$$\begin{aligned}
&\|F_{A^\Sigma} + [\Phi \wedge \Phi^*]\|^2 + \tfrac{1}{4}\|[\phi, \phi^*]\|^2 + \tfrac{1}{4}\|[\gamma, \gamma^*]\|^2 + \\
&+ \int_\Sigma \langle i\Lambda [\Phi \wedge \Phi^*], [\phi, \phi^*]\rangle + \int_\Sigma \langle i\Lambda [\Phi \wedge \Phi^*], [\gamma, \gamma^*]\rangle + \tfrac{1}{2}\int_\Sigma \langle [\phi, \phi^*], [\gamma, \gamma^*]\rangle + \\
&+ \int_\Sigma \langle i\Lambda F_{A^\Sigma} \cdot [\phi, \phi^*]\rangle + \int_\Sigma \langle i\Lambda F_{A^\Sigma} \cdot [\gamma, \gamma^*]\rangle + 2\left\|\bar\partial_{A^\Sigma}\Phi\right\|^2 + \left\|\bar\partial_{A^\Sigma}\phi\right\|^2 + \left\|\bar\partial_{A^\Sigma}\gamma\right\|^2 + \\
&+ \tfrac{1}{2}\|[\gamma^*, \phi^*]\|^2 + \|[\phi^*, \Phi^*]\|^2 + \|[\Phi^*, \gamma^*]\|^2 + \\
&- 2\left(\int_\Sigma \left\langle i\Lambda\bar\partial_{A^\Sigma}\Phi, [\gamma^*, \phi^*]\right\rangle + \int_\Sigma \left\langle \bar\partial_{A^\Sigma}\phi, [\Phi^*, \gamma^*]\right\rangle + \int_\Sigma \left\langle \bar\partial_{A^\Sigma}\gamma, [\phi^*, \Phi^*]\right\rangle\right).
\end{aligned}$$

A solution to the reduced Vafa-Witten equations is equivalent to a zero of this expression. We will now rewrite most of these terms.

First, we have the algebraic fact

$$\langle [\phi, \phi^*], [\gamma, \gamma^*]\rangle = |[\phi, \gamma^*]|^2 - |[\phi^*, \gamma^*]|^2 .$$

Similarly,

$$\begin{aligned}
\langle i\Lambda [\Phi \wedge \Phi^*], [\phi, \phi^*]\rangle &= 2\left\langle \left[\frac{\Phi}{dz}, \frac{\Phi^*}{d\bar z}\right], [\phi, \phi^*]\right\rangle \\
&= 2\left|\left[\frac{\Phi}{dz}, \phi^*\right]\right|^2 - 2\left|\left[\frac{\Phi^*}{d\bar z}, \phi^*\right]\right|^2 \\
&= |[\Phi, \phi^*]|^2 - |[\Phi^*, \phi^*]|^2 .
\end{aligned}$$

By (7.1),

$$\int_\Sigma \langle i\Lambda F_{A^\Sigma} \cdot [\phi, \phi^*]\rangle = \|\partial_{A^\Sigma}\phi\|^2 - \left\|\bar\partial_{A^\Sigma}\phi\right\|^2.$$

Finally, for this next equation, let Tr denote the operator characterized by $\mathrm{Tr}(AB^*) = \langle A, B\rangle$. Then

$$\begin{aligned}
\Lambda d\mathrm{Tr}(i\Phi\,[\phi,\gamma]) &= i\Lambda\mathrm{Tr}\left(\left(\bar\partial_{A^\Sigma}\Phi\right)[\phi,\gamma] - \Phi\left[\bar\partial_{A^\Sigma}\phi,\gamma\right] - \Phi\left[\phi,\bar\partial_{A^\Sigma}\gamma\right]\right)\\
&= i\Lambda\mathrm{Tr}\left(\left(\bar\partial_{A^\Sigma}\Phi\right)[\phi,\gamma] + \left(\bar\partial_{A^\Sigma}\phi\right)[\gamma,\Phi] + \left(\bar\partial_{A^\Sigma}\gamma\right)[\Phi,\phi]\right)\\
&= i\Lambda\mathrm{Tr}\left(\left(\bar\partial_{A^\Sigma}\Phi\right)[\gamma^*,\phi^*]^* + \left(\bar\partial_{A^\Sigma}\phi\right)[\Phi^*,\gamma^*]^* + \left(\bar\partial_{A^\Sigma}\gamma\right)[\phi^*,\Phi^*]^*\right)\\
&= \left\langle i\Lambda\left(\bar\partial_{A^\Sigma}\Phi\right)\cdot[\gamma^*,\phi^*]\right\rangle + \left\langle\left(\bar\partial_{A^\Sigma}\phi\right)\cdot[\Phi^*,\gamma^*]\right\rangle + \left\langle\left(\bar\partial_{A^\Sigma}\gamma\right)\cdot[\phi^*,\Phi^*]\right\rangle,
\end{aligned}$$

where for the last line, we used the fact $i\,dz\wedge d\bar z = \omega\,(dz\cdot d\bar z)$ to convert from the wedge product to the inner product. Hence

$$\int_\Sigma \left\langle i\Lambda\bar\partial_{A^\Sigma}\Phi, [\gamma^*,\phi^*]\right\rangle + \int_\Sigma \left\langle\bar\partial_{A^\Sigma}\phi, [\Phi^*,\gamma^*]\right\rangle + \int_\Sigma \left\langle\bar\partial_{A^\Sigma}\gamma, [\phi^*,\Phi^*]\right\rangle = 0.$$

The sum of norms simplifies to

$$\begin{aligned}
&\|F_{A^\Sigma} + [\Phi\wedge\Phi^*]\|^2 + 2\left\|\bar\partial_{A^\Sigma}\Phi\right\|^2 + \tfrac14\|[\phi,\phi^*]\|^2 + \tfrac14\|[\gamma,\gamma^*]\|^2 +\\
&+ \|[\Phi,\phi^*]\|^2 + \|[\Phi,\gamma^*]\|^2 + \tfrac12\|[\phi,\gamma^*]\|^2 + \|\partial_{A^\Sigma}\phi\|^2 + \|\partial_{A^\Sigma}\gamma\|^2.
\end{aligned}$$

Since this sum of positive terms vanishes on a solution, each term must vanish individually. $\square$

Chapter 9

Gluing

In order to investigate the dependence of the supposed Vafa-Witten invariants on the choice of compactification, we initiate a study of the Uhlenbeck boundary of the Vafa-Witten moduli space. This is a work-in-progress, with the goal of developing quadratic models in the spirit of [Tau84] and [Don86].

As we vary through a one-parameter family of metrics g_t, the topology of the ASD moduli space may change. If this change occurs away from the Uhlenbeck boundary, then it is described as the neighborhood of a singular connection A in the t-parameterized moduli space. Such a singularity can be modeled upon the zero set of a quadratic expression, provided that the expression is non-degenerate.

When the change occurs on the Uhlenbeck boundary, the singularity can be described by gluing techniques. These gluing techniques provide both a grafting map and an obstruction map. The grafting parameters augment the tangent space parameters H^1_A, while the obstruction map augments the obstruction H^2_A. Upon adding an extra term ([Tau84, eq. (1.7)] or more generally [Don86, eq. (5.3)]), the previous quadratic model extends to a description of the corresponding singularity as it appears in the Uhlenbeck boundary of the higher-level moduli spaces.

Our (incomplete) goal is to extend these techniques from the parameterized ASD moduli space to the parameterized Vafa-Witten moduli space. In Chapter 5, we already derived a quadratic model (5.4) which describes neighborhoods for the uncompactified moduli space, i.e. the region away from the Uhlenbeck boundary. To make matters simpler, we will focus on describing the special case of a neighborhood in the Vafa-Witten moduli space of an ASD connection, i.e. the region of $\mathcal{M}_{\mathrm{VW}}$ where B is very small. In this case, the quadratic model reduces to (5.5) and (5.6).

To extend this model to cover corresponding singularities on the Uhlenbeck boundary, it remains to compute the extra terms which arise from gluing. After describing some standard constructions in Section 9.1, we describe in Chapter **??** what we expect from the quadratic model at the Uhlenbeck boundary.

We also comment that similar techniques might provide insight into the region of the moduli space where B blows up. In particular, it would be interesting to explore what happens when we graft instantons onto harmonic self-dual two-forms.

9.1 Grafting instantons

The standard ASD instanton A_λ of width λ over flat quaternionic space $\mathbb{H} = \mathbb{R}^4$ is given by the connection matrix

$$A_\lambda := \frac{\operatorname{Im}(\overline{x}\,dx)}{\lambda^2 + |x|^2},$$

satisfies the Coulomb gauge condition $d^* A_\lambda = 0$, and has curvature

$$F_{A_\lambda} = \frac{\lambda^2\,\overline{dx} \wedge dx}{\left(\lambda^2 + |x|^2\right)^2},$$

where

$$\begin{aligned}
\operatorname{Im}(\overline{x}\,dx) &= \left(-x^1e^0 + x^0e^1 + x^3e^2 - x^2e^3\right)\mathbf{i} + \left(-x^2e^0 - x^3e^1 + x^0e^2 + x^1e^3\right)\mathbf{j} + \\
&\quad + \left(-x^3e^0 + x^2e^1 - x^1e^2 + x^0e^3\right)\mathbf{k},\\
\operatorname{Im}(dx\,\overline{x}) &= \left(-x^1e^0 + x^0e^1 - x^3e^2 + x^2e^3\right)\mathbf{i} + \left(-x^2e^0 + x^3e^1 + x^0e^2 - x^1e^3\right)\mathbf{j} + \\
&\quad + \left(-x^3e^0 - x^2e^1 + x^1e^2 + x^0e^3\right)\mathbf{k},\\
\tfrac{1}{2}\overline{dx} \wedge dx &= \left(e^{01} + e^{23}\right)\mathbf{i} + \left(e^{02} + e^{31}\right)\mathbf{j} + \left(e^{03} + e^{12}\right)\mathbf{k},
\end{aligned}$$

cf. [FU90, p. 88] and [Tau84].

We define the gauge transformation u over $\mathbb{H}\setminus\{0\}$ by

$$\mathsf{u}(x) := \frac{x}{|x|} \in \operatorname{Sp}(1).$$

Note the the identities

$$\begin{aligned}
\mathsf{u}A_\lambda\mathsf{u}^{-1} &= \frac{\operatorname{Im}(dx\,\overline{x})}{\lambda^2 + |x|^2},\\
-d\mathsf{u}\,\mathsf{u}^{-1} &= -\frac{\operatorname{Im}(dx\,\overline{x})}{|x|^2}.
\end{aligned}$$

The gauge transformation u acts on A_λ, which we write as $\mathsf{u} \curlywedge A_\lambda$. We define

$$A'_\lambda := \mathsf{u} \curlywedge A_\lambda = \mathsf{u}A_\lambda\mathsf{u}^{-1} - d\mathsf{u}\,\mathsf{u}^{-1} = -\frac{\lambda^2\operatorname{Im}(dx\,\overline{x})}{|x|^2\left(\lambda^2 + |x|^2\right)}.$$

This satisfies $d^* A'_\lambda = 0$, and

$$F_{A'_\lambda} = \mathsf{u} \curlywedge F_{A'_\lambda} = \mathsf{u}F_{A_\lambda}\mathsf{u}^{-1} = \frac{\lambda^2}{|x|^2}\frac{\overline{(dx\,\overline{x})} \wedge (dx\,\overline{x})}{\left(\lambda^2 + |x|^2\right)^2}.$$

The pointwise norms of these connections A_λ and A'_λ are

$$|A_\lambda| = \frac{\sqrt{3}\,|x|}{\lambda^2 + |x|^2}, \qquad |A'_\lambda| = \frac{\sqrt{3}\lambda^2}{|x|\left(\lambda^2 + |x|^2\right)}.$$

Notice how when x is small, A_λ is small while A'_λ is large. Conversely when x is large, A'_λ decays as $|x|^{-3}$ while A_λ decays only as $|x|^{-1}$. Heuristically, for any connection matrix with localized curvature, we should expect no steeper than inverse cubic decay, in accordance with the Green's function of a first-order operator in four dimensions.

Another important property of the standard instanton is that the orientation-preserving involution

$$x \mapsto \frac{\overline{x}}{|x|^2}$$

acting by pullback on the domain exchanges

$$A'_\lambda \leftrightarrow A_{1/\lambda}.$$

For any rotation $r \in \mathrm{SO}(\mathbb{H})$, we can find $r^-, r^+ \in \mathrm{Sp}(1)$ (unique up to common sign) such that

$$r(x) = r^- x \overline{r^+},$$

where we recall that $\overline{r^+} = (r^+)^{-1}$ for $r^+ \in \mathrm{Sp}(1)$.

Now we examine the pullback of connections over $\mathbb{H}$ by such rotations r. Our particular gauge transformation u intertwines rotations by the rule

$$r^*(\mathsf{u} \curlywedge A) = (r^- \mathsf{u} \overline{r^+}) \curlywedge r^*(A),$$

for any connection A. In particular, A_λ and A'_λ transform as

$$\begin{aligned} r^*(A_\lambda) &= r^+ A_\lambda \overline{r^+} = r^+ \curlywedge A_\lambda, \\ r^*(A'_\lambda) &= r^- A'_\lambda \overline{r^-} = r^- \curlywedge A'_\lambda. \end{aligned}$$

Given a principal $\mathrm{Sp}(1)$ bundle $P \to X$ equipped with a background connection A, we define a grafting map as follows. Pick a point $x_0 \in X$, an oriented frame of $f \in T_{x_0}X$, and a point $p \in P_{x_0}$ in the fiber over x_0. Next, consider a geodesic coordinate chart based at f, and the local trivialization τ_p for P induced by the radial gauge of A based at p. Upon removing the fiber of P at x and attaching a new chart to τ_p via the automorphism u, we get a new bundle P' with $c_2(P') = 1 + c_2(P)$. Connections on this new bundle are specified near x by a pair

$$(A^{\mathrm{out}}, A^{\mathrm{in}}) \text{ satisfying } A^{\mathrm{out}} = \mathsf{u} \curlywedge A^{\mathrm{in}}.$$

For sufficiently small $\lambda_{\max}$, given any $\lambda \in (0, \lambda_{\max})$ and families of cutoff functions $\beta_\lambda^{\text{out}}$ and $\beta_\lambda^{\text{in}}$ with

$$0 = \beta_\lambda^{\text{out}}(0) = \beta_\lambda^{\text{in}}(\lambda_{\max}),$$
$$1 = \beta_\lambda^{\text{out}}(\lambda_{\max}) = \beta_\lambda^{\text{in}}(0),$$

we define the grafted connection to be given in our geodesic coordinate chart by

$$(\beta_\lambda^{\text{out}} A^{\tau_p} + \beta_\lambda^{\text{in}} A'_\lambda, \beta_\lambda^{\text{out}}(\mathsf{u}^{-1} \ltimes A^{\tau_p}) + \beta_\lambda^{\text{in}} A_\lambda),$$

and equal to the background connection A elsewhere.

Fixing the cutoff functions and the background connection, the choice of parameters is given by our basepoint x, our oriented frame of $T_x X$, and our frame $p \in P$, and the parameter λ. This is a total of fourteen dimensions, parameterized locally by

$$\mathbb{R}^4 \times \mathrm{SO}(4) \times \mathrm{Sp}(1) \times (0, \lambda_{\max}).$$

This parameter space reduces due to the symmetries of the instanton. Rewriting $\mathrm{SO}(4) = \mathrm{Sp}(1)^- \times_{\mathbb{Z}_2} \mathrm{Sp}(1)^+$, the symmetry subgroup of the standard instanton is $\mathrm{Sp}(1)^- \times \operatorname{diag}(\mathrm{Sp}(1)^+, \mathrm{Sp}(1))$. Accounting for this symmetry, the effective parameter space is eight-dimensional, given by

$$\mathbb{R}^4 \times \mathrm{SO}(3) \times (0, \lambda_{\max}).$$

This copy of $\mathrm{SO}(3)$ is described invariantly by

$$\operatorname{Isom}(\Lambda^{2,+} T^*_{x_0} X, (\mathfrak{g}_P)_{x_0}).$$

We define $N \to X$ to be the $\mathrm{SO}(3) \times (0, \lambda_{\max})$-bundle

$$N := \operatorname{Isom}(\Lambda^{2,+} T^* X, \mathfrak{g}_P). \times (0, \lambda_{\max}).$$

The total space of N is eight-dimensional, and it describes our grafting parameters.

If A is ASD, and if the cutoff functions β are chosen appropriately, then the grafted connections are approximately ASD in the sense that small perturbations often make them exactly ASD.

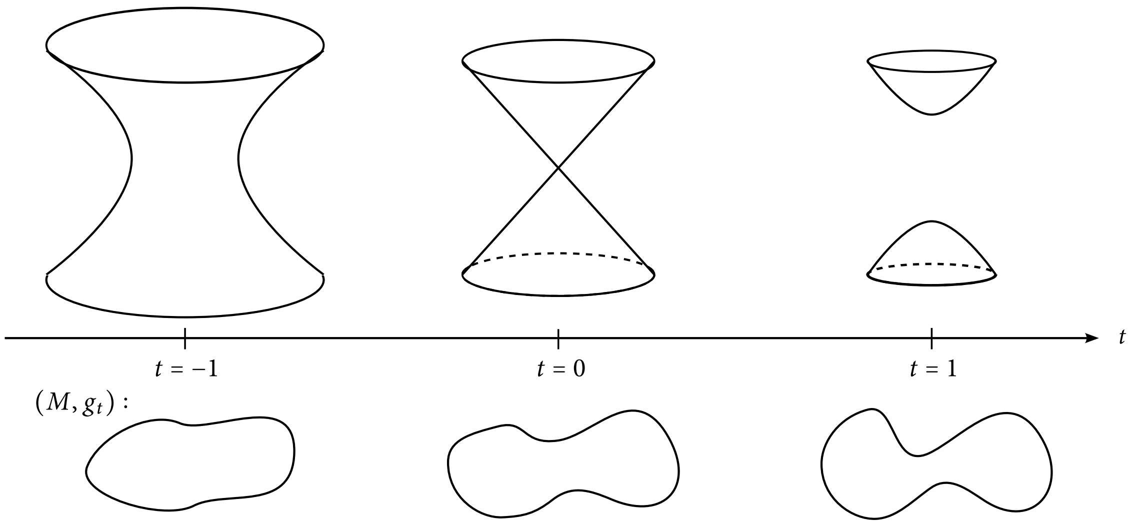
9.2 The gluing story

Suppose $d := \dim \mathcal{M}_{\text{ASD}}$, and that $A \in \mathcal{M}_{\text{ASD}}$ is an irreducible but singular point such that the Kuranishi cohomology has $\dim H_A^2 = 1$ with $H_A^2 = \operatorname{span}(\hat{b})$. Note that since the index must be preserved, $\dim H_A^1 = d + 1$.

A model for the neighborhood of $A \in \mathcal{M}_{\text{ASD}}$ is given, as a subset of H_A^1, by the zero set of a quadratic "obstruction map"

$$q(a) = \left\langle \tfrac{1}{2} [a \wedge a]^+ \cdot \hat{b} \right\rangle, \text{ for } a \in H_A^1,$$

Figure 9-1: Cone model about a nondegenerate singular instanton, as the metric is perturbed.



and this is a singular cone.

By the metric transversality theorem for the $\mathrm{SU}(2)$ ASD equations, there exists some perturbation of the conformal structure $m \in \mathrm{Hom}(\Lambda^{2,-}, \Lambda^{2,+})$ such that $\left\langle mF_A^- \cdot \hat{b}\right\rangle \neq 0$. Consider a family of metrics g_t such that $\dot{g}_0$ induces the perturbation m. Then the local model for the parameterized moduli space is

$$\left\langle \left(\tfrac{1}{2}\left[a \wedge a\right]^+ + tmF_A^-\right) \cdot \hat{b}\right\rangle = 0.$$

Assuming that the quadratic form given on H_A^1 by $\left\langle \tfrac{1}{2}\left[a \wedge a\right]^+ \cdot \hat{b}\right\rangle$ is nondegenerate, this describes a standard surgery at $A \in \mathcal{M}_{\mathrm{ASD}}$.

If $\mathcal{M}_{\mathrm{ASD}} = \mathcal{M}_{\mathrm{ASD},k}$ has instanton number k, then the next highest moduli space $\mathcal{M}_{\mathrm{ASD},k+1}$ has an Uhlenbeck compactification

$$\mathcal{M}_{\mathrm{ASD},k+1} \cup (X \times \mathcal{M}_{\mathrm{ASD},k}),$$

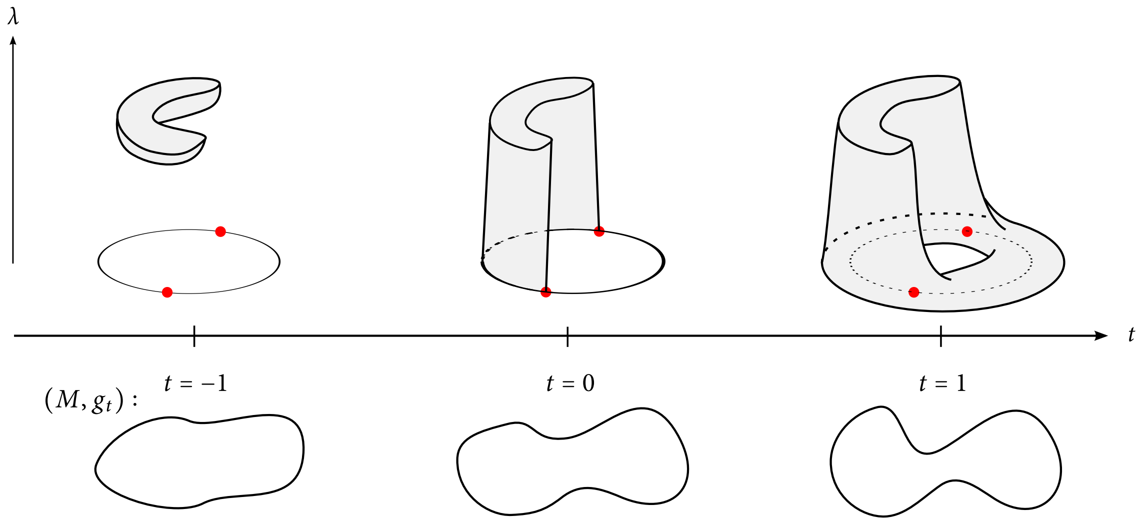
where for simplicity we assume that $\mathcal{M}_{\mathrm{ASD},k-1} = \varnothing$. According to [Don86], the end of this $d+8$-dimensional moduli space is described by a background connection $A \in \mathcal{M}_{\mathrm{ASD},k}$ and an eight-dimensional pair of gluing parameters (x, q^2), with $x \in X$ and $q^2 \in \mathbb{R}\mathrm{SO}((\mathfrak{sp}(1)_P)_x, (\Lambda^{2,+})_x)$. (We can think of q as a quaternion, so that q^2 is in the orbifold $\mathbb{H}/(q^2 \sim (-q)^2)$, which is a cone on $\mathbb{RP}^3$.)

If the background connection A is obstructed by $\hat{b} \in H_A^2$, then we get a quadratic model for the end. Again we consider the t-parameterized moduli space, and the local model is

$$\left\langle \left(\tfrac{1}{2}\left[a \wedge a\right]^+ + \delta_x q^2 + tmF_A^-\right) \cdot \hat{b}\right\rangle = 0.$$

Here the notation $\delta_x q^2$ simply means that we pair q^2 with $\hat{b}$ at the point x. (This is simply *notation,* just to emphasize that the vector q^2 dual to $\hat{b}$ is supported at the point x, like a delta function.) This model allows us to explicitly model the change in topology near the end. For illustrative purposes, we reduce the eight-dimensional picture to a two-dimensional one by considering $X = S^1$ and

Figure 9-2: Cone model about a nondegenerate singular instanton, as the metric is perturbed.



$q^2 = \lambda^2 \in \mathbb{R}^{\geq 0}$. Furthermore, consider the case $d = 0$ so $H^1_A = \operatorname{span}(a)$. Then as a function of x,

$$\left\langle \delta_x \lambda^2 \cdot \hat{b} \right\rangle$$

may change sign. This sign determines whether instantons appear over x when t is either positive or negative. We get the following sort of picture describing the change in topology of $\mathcal{M}_{\mathrm{ASD},k+1}$:

This diagram illustrates a cup morphing into a pair-of-pants, where the nodes of $\left\langle \delta_x \lambda^2 \cdot \hat{b} \right\rangle$ are indicated by the two dots.

Now consider passing from the ASD equations to the Vafa-Witten equations. Since the Vafa-Witten equations are the ASD equations plus their adjoint, the Kuranishi cohomology of the Vafa-Witten complex is given in terms of the ASD Kuranishi cohomology by

$$H^1_{(0,A,0)} \cong H^2_{(0,A,0)} \cong H^1_A \oplus H^2_A.$$

The local model as determined in 5.5 is essentially equivalent to

$$\begin{aligned} \left\langle \left(\tfrac{1}{2} [a \wedge a]^+ + \tfrac{1}{8} [b \,.\, b] - tmF^-_A \right) \cdot \hat{b} \right\rangle &= 0, \\ \left\langle [a \,.\, b] \cdot \hat{a} \right\rangle &= 0. \end{aligned}$$

When the second equation is nondegenerate, it has the interpretation: "either $a = 0$ or $b = 0$." For the $b = 0$ case, we recover the original $\mathcal{M}_{\mathrm{ASD}}$. For the $a = 0$ case, if $\dim H^2_A = 1$, then we get $b = \sqrt{\xi t}$ for some constant ξ. Thus we see either the creation or destruction (depending on the sign of ξ) of a pair of extra points in $\mathcal{M}_{\mathrm{VW}}$, and these extra points carry the topology which was created/destroyed in $\mathcal{M}_{\mathrm{ASD}}$.

Passing to $\mathcal{M}_{\mathrm{VW},k+1}$, we expect a similar obstruction arising from $H^2_{(0,A,0)}$ given by

$$\begin{aligned}\left\langle\left(\tfrac{1}{2}\left[a\wedge a\right]^+ + \tfrac{1}{8}\left[b \mathbin{\scriptstyle\blacksquare} b\right] - tmF_A^- + \delta_x q^2\right)\cdot\hat{b}\right\rangle = 0,\\ \left\langle\left[a \mathbin{\scriptstyle\blacksquare} b\right]\cdot\hat{a}\right\rangle = 0.\end{aligned}$$

However, unlike the ASD equations, the standard instanton I is Vafa-Witten-obstructed by $H^2_{(0,I,0)} \cong H^1_I$, which is eight-dimensional. (Eight dimensions rather than five since when we glue, we must choose a "trivialization at infinity" which gives an extra $\mathrm{SO}(3)$.) Thus we expect an extra eight constraints of the form

$$\left\langle\delta_x\left[b \mathbin{\scriptstyle\blacksquare} \iota\right]\cdot\hat{a}\right\rangle = 0$$

for $\iota\in H^1_I$. In the generic case, the solutions with $b\neq 0$ should consist of a discrete set of points. These extra points should say something very interesting about whether or not a chosen compactification leads to a well-defined invariant.

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Appendix A

Notation and conventions

The goal of this appendix is to establish consistent conventions to facilitate precise computations.

A.1 Linear algebra

A.1.1 Components of linear maps

Here we describe some details of our index notation. We use the Einstein summation convention, keeping track of the left-to-right order of indices.

Let V be a real vector space with basis $\{e_i\}_{i=1}^n$ and corresponding dual basis $\{e^i\}_{i=1}^n$. A vector $v \in V$ has components $v = e_i v^i$, and a covector $\alpha \in V^*$ has components $\alpha = e^i \alpha_i$. There is a natural symmetric duality pairing between V and V^* given by

$$\alpha \cdot v = v \cdot \alpha = v_i \alpha^i.$$

Let W be another vector space with basis $\{f_j\}_{j=1}^m$ and dual basis $\{f^j\}_{j=1}^m$. Any $L \in \operatorname{Hom}(V, W)$ has components $L(e_i) = f_j L^j{}_i$ so that

$$w = Lv \iff f_j w^j = f_j L^j{}_i v^i \iff w^j = L^j{}_i v^i.$$

Similarly, the dual map $L^* \in \operatorname{Hom}(W^*, V^*)$ has components $L^*(f^i) = e^j L^*{}_j{}^i$. It follows that $L^*{}_j{}^i = L^i{}_j$, and

$$\alpha = L^* \beta \iff e^i \alpha_i = e^i L^*{}_i{}^j \beta_j \iff \alpha_i = L^j{}_i \beta_j.$$

If V comes equipped with a Euclidean metric $g = g_{ij}$, then we may view g either as a nondegenerate symmetric bilinear form $g \in \operatorname{Sym}^2(V^*)$, or as a map $g \in \operatorname{Hom}(V, V^*)$. We take the latter view, which is more convenient for our purposes. The "symmetry" condition on g means that $g^* = g$.

We denote the components of the inverse by $g^{-1} = g^{ij}$ so that $g^{ik} g_{kj} = \delta^i_j$. In components, we have the lowering and raising operators

$$g(v) = e^i g_{ij} v^j, \qquad g^{-1}(\alpha) = e_i g^{ij} \alpha_j.$$

We implicitly use the metric to extend the duality pairing

$$\begin{aligned} v \cdot w &:= g(v) \cdot w = v \cdot g(w) && \text{for } v, w \in V, \\ \alpha \cdot \beta &:= g^{-1}(\alpha) \cdot \beta = \alpha \cdot g^{-1}(\beta) && \text{for } \alpha, \beta \in V^*. \end{aligned}$$

In components, we implicitly use the metric to raise and lower indices. For example, if $v \in V$, then

$$v_i := g_{ij} v^j.$$

This convention does not apply to the basis vectors themselves, since $e_i \neq g_{ij} e^j$ unless $\{e_i\}$ is orthonormal.

While the metric allows us to raise and lower indices at will, it is essential to keep track of the left-to-right order of tensor indices, since $L_{ij} \neq L_{ji}$ unless L is symmetric.

A.1.2 Representations on the dual space

This subsection explains the reasoning behind the convention described in Remark A.1.2.

The dual left representation $\mathrm{GL}(V)$ on V^* is given by

$$\begin{aligned} \mathrm{GL}(V) &\longrightarrow \mathrm{GL}(V^*), \\ L &\mapsto (L^*)^{-1}. \end{aligned} \tag{A.1}$$

The corresponding Lie algebra representation is

$$\begin{aligned} \mathfrak{gl}(V) &\longrightarrow \mathfrak{gl}(V^*), \\ L &\mapsto -L^*. \end{aligned} \tag{A.2}$$

If V comes equipped with a Euclidean metric g, then for any $L \in \mathrm{End}(V)$, we define the *metric adjoint* $L^T \in \mathrm{End}(V)$, which is characterized by

$$Lv \cdot w = v \cdot L^T w,$$

and given explicitly by

$$L^T = g^{-1} L^* g.$$

Remark A.1.1. By widespread abuse of notation, L^T is typically denoted by L^* in conflict with the notation for the dual map. For example, the standard notation for the metric adjoint of the exterior derivative d (over a closed manifold) is d^* rather than d^T. To increase readability, we abandon the notation L^T outside of this subsection.

In the case where V comes equipped with a Euclidean metric, we have an alternative to the representations (A.1) and (A.2), given by the *metric adjoint representation*

$$\begin{aligned} \mathrm{End}(V) &\longrightarrow \mathrm{End}(V^*), \\ L &\mapsto (L^*)^T = gLg^{-1}. \end{aligned} \tag{A.3}$$

This is a representation of associative algebras. It agrees with (A.1) and (A.2) respectively when restricted to orthogonal and antisymmetric endomorphisms

$$\begin{aligned} \mathrm{O}(V) &= \left\{ L \in \mathrm{End}(V) | L^T = L^{-1} \right\}, \\ \mathfrak{o}(V) &= \left\{ L \in \mathrm{End}(V) | L^T = -L \right\}. \end{aligned}$$

Remark A.1.2. The representations (A.2) and (A.3) disagree in general. For symmetric endomorphisms, they differ by a sign. Since all vector spaces of interest will have metrics, we abandon (A.2) and rely entirely on the metric adjoint representation (A.3), and incorporate it into our notation. For $L \in \mathrm{End}(V)$ and $\alpha \in V^*$, we define

$$L\alpha := (L^*)^T \alpha. \tag{A.4}$$

In components, we write this in the following possible ways:

$$(L\alpha)_i := g_{ij} L^j{}_k g^{k\ell} \alpha_\ell = L_i{}^j \alpha_j = L_{ij} \alpha_j.$$

A.1.3 The exterior algebra

We view $\Lambda^\bullet V^*$ as the free graded-commutative $\mathbb{R}$-algebra with identity, generated in degree one by V^*. If $\alpha \in \Lambda^p V^*$, then we write the components of α in the following ways:

$$\alpha = \frac{1}{p!} \alpha_{I_1 \cdots I_p} e^{I_1} \wedge \cdots \wedge e^{I_p} = \frac{1}{p!} \alpha_{I_1 \cdots I_p} e^{I_1 \cdots I_p} = \frac{1}{p!} \alpha_I e^I = \sum_{I \text{ increasing}} \alpha_I e^I. \tag{A.5}$$

If α is homogeneous, we let $|\alpha|$ denote the degree of α, and $|I|$ the length of the multiindex I.

The duality pairing on $V^* \otimes V$ extends to $\Lambda^\bullet V^* \otimes \Lambda^\bullet V$ by the rule

$$e^I \cdot e_J = \begin{cases} \varepsilon^I_J & \text{if } |I| = |J|, \\ 0 & \text{otherwise,} \end{cases}$$

where ε denotes the antisymmetric tensor which gives the relative sign of two permutations.

We define metric raising and lowering operators on a basis by

$$\begin{aligned} g(e_I) &:= g(e_{I_1}) \wedge \cdots \wedge g(e_{I_k}), \\ g^{-1}(e^I) &:= g^{-1}(e^{I_1}) \wedge \cdots \wedge g^{-1}(e^{I_k}). \end{aligned}$$

The duality pairing on $\Lambda^\bullet V^* \otimes \Lambda^\bullet V$ extends to metric pairings on $\Lambda^\bullet V^* \otimes \Lambda^\bullet V^*$ and $\Lambda^\bullet V \otimes \Lambda^\bullet V$ given by

$$\alpha \cdot \beta := \alpha \cdot g^{-1}(\beta), \qquad v \cdot w := g(v) \cdot w.$$

A.1.4 The fermionic oscillator algebra

For a finite-dimensional vector space V, the fermionic oscillator algebra $\Theta^\bullet(V)$ conveniently describes endomorphisms of $\Lambda^\bullet V^*$. Unlike the Clifford algebra A.1.5, the fermionic oscillator algebra does not involve a metric. The oscillator algebra $\Theta^\bullet(V)$ is a $\mathbb{Z}$-graded $\mathbb{R}$-algebra generated by the image of two linear maps $\iota : V \hookrightarrow \Theta^{-1}(V)$ and $\epsilon : V^* \hookrightarrow \Theta^1(V)$ and characterized by the relations

$$\begin{aligned}
&\forall v, w \in V,\ \ \forall \alpha, \beta \in V^*,\\
&\iota(v)\iota(w) + \iota(w)\iota(v) = 0,\\
&\epsilon(\alpha)\epsilon(\beta) + \epsilon(\beta)\epsilon(\alpha) = 0,\\
&\epsilon(\alpha)\iota(v) + \iota(v)\epsilon(\alpha) = \alpha \cdot v.
\end{aligned}$$

If V has basis $\{e_i\}$ and dual basis $\{e^i\}$, then we use the abbreviations

$$\begin{aligned}
\iota_i &:= \iota(e_i), \quad & \iota_I &:= \iota_{I_1} \cdots \iota_{I_k},\\
\epsilon^i &:= \epsilon(e^i), \quad & \epsilon^I &:= \epsilon^{I_1} \cdots \epsilon^{I_k}.
\end{aligned}$$

There is an action $\Theta^\bullet(V) \to \mathrm{End}(\Lambda^\bullet(V^*))$ generated by the standard contraction and wedge maps

$$\begin{aligned}
\epsilon^i &: e^{I_1\cdots I_k} \mapsto e^i \wedge e^{I_1\cdots I_k}, \qquad\qquad (A.6)\\
\iota_i &: e^{I_1\cdots I_k} \mapsto \sum_{j=1}^{k} (-1)^{j-1} \left(e^{I_j} \cdot e_i\right) e^{I_1\cdots I_{j-1}} \wedge e^{I_{j+1}\cdots I_k}.
\end{aligned}$$

Note that $\epsilon(\alpha)$ acts as a multiplication, while $\iota(v)$ acts as a graded derivation. Specifically, that $\epsilon(v)$ acts a graded derivation means that for any $\alpha, \beta \in \wedge^\bullet V^*$,

$$\epsilon(v)(\alpha \wedge \beta) = (\epsilon(v)\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge (\epsilon(v)\beta). \tag{A.7}$$

The *normal-ordered monomials* are monomials in $\Theta^\bullet(V)$ of the form $\epsilon^I \iota_J$ for increasing multi-indices I and J. The normal-ordered monomials provide a vector space basis for $\Theta^\bullet(V)$. Note that $\dim \Theta^k(V) = \begin{pmatrix} 2\dim V \\ k + \dim V \end{pmatrix}$, and

$$\dim \Theta^\bullet(V) = \sum_{k=-n}^{n} \dim \Theta^k(V) = 4^{\dim V} = \dim \mathrm{End}(\Lambda^\bullet(V^*)).$$

The normal ordered monomials in $\Theta^\bullet(V)$ are easily seen to act linearly independently on $\Lambda^\bullet(V^*)$. Therefore, the action map $\Theta^\bullet(V) \to \mathrm{End}(\Lambda^\bullet(V^*))$ is injective. Since the dimensions coincide, this action map must be an isomorphism.

There are two important duality relations for $\Theta^\bullet(V)$. First, recall that the dual space $\mathrm{End}(\Lambda^\bullet(V^*))^*$ is canonically isomorphic to $\mathrm{End}(\Lambda^\bullet(V))$. There is a natural dual action

$$\Theta^\bullet(V) \to \mathrm{End}(\Lambda^\bullet(V))$$

for which $\iota(v)$ acts on a multivector as wedge, and $\epsilon(\alpha)$ acts on a multivector as contraction. These two representations are adjoint in the sense that for $v \in \Lambda^\bullet(V)$, $w \in V$, $\alpha \in \Lambda^\bullet(V^*)$, $\beta \in V^*$,

$$\iota(w)\alpha \cdot v = \alpha \cdot \iota(w)v \text{ and } \epsilon(\beta)\alpha \cdot v = \alpha \cdot \epsilon(\beta)v. \tag{A.8}$$

In contrast, $\iota(v)$ is metric-adjoint to $\epsilon(g(v))$ in the sense that for $v \in V$, $\alpha, \beta \in \Lambda^\bullet(V^*)$,

$$\iota(v)\alpha \cdot \beta = \alpha \cdot \epsilon(g(v))\beta.$$

We now assume that V comes equipped with a metric so that there is a metric adjoint action of $\mathrm{End}(V)$ on V^* (A.4). We extend

this action to a derivation on all of $\Lambda^\bullet V^*$. It's easy to check that the embedding

$$\mathrm{End}(V) \hookrightarrow \mathrm{Der}(\Lambda^\bullet V^*) \subset \Theta^\bullet(V)$$

is given by

Definition A.1.3. For a Riemannian vector space V, the *standard action* of a rank two tensor $L = L_{ij}$ acting on $\alpha \in \Lambda^\bullet V^*$ as a derivation is given by

$$L\alpha := L_{ij}\epsilon^i \iota_j \alpha. \tag{A.9}$$

A.1.5 The Clifford algebra

If V has a metric g, then the *Clifford algebra* $\mathrm{Cl}^\bullet(V, g)$ is the $\mathbb{Z}_2$-graded associative algebra generated by $\gamma : V \hookrightarrow \mathrm{Cl}^1(V, g)$ subject to the relation

$$\gamma(v)\gamma(w) + \gamma(w)\gamma(v) + 2v \cdot w = 0.$$

There is a homomorphism $\mathrm{Cl}^\bullet(V) \to \Theta^\bullet(V)$ generated by

$$\gamma(v) \mapsto \epsilon(g(v)) - \iota(v). \tag{A.10}$$

A.1.6 The $\mathbin{\text{.}}$ product on $\Lambda^\bullet V^*$

When V comes equipped with a metric g, we define the following product on $\Lambda^\bullet V^*$.

Definition A.1.4. For any $\alpha \in \Lambda^{|\alpha|} V^*$ and $\beta \in \Lambda^{|\beta|} V^*$,

$$\alpha \mathbin{\text{.}} \beta := (-1)^{|\alpha|-1}(a_i\alpha) \wedge (a_i\beta) \in \Lambda^{|\alpha|+|\beta|-2} V^*, \tag{A.11}$$

where $a_i\alpha$ was defined by (A.6). The $(-1)^{|\alpha|-1}$ factor yields the desired rule (A.12).

Next, we fix an identification between antisymmetric transformations $L \in \mathfrak{o}(V)$ and $\Lambda^2 V^*$.

Definition A.1.5. The isomorphism $\theta : \mathfrak{o}(V) \to \Lambda^2 V^*$ identifies $L \in \mathfrak{o}(V)$ with the element of $\Lambda^2 V^*$ having the same components. Specifically, by (A.5) we have $\theta(X) := \frac{1}{2} X_{ij} e^{ij}$.

For example, if $n = 2$ and L is given by the matrix

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then $\theta(L) = e^{12}$.

Proposition A.1.6. *If g is positive-definite, and if $\{e^i\}$ is an orthonormal coframe, then for all $X, Y \in \mathfrak{o}(V)$, and all $\alpha, \beta, \gamma \in \Lambda^\bullet V^*$ of homogeneous degree, the following identities hold:*

$$e^{1\cdots p_1} \centerdot e^{p_1\cdots(p_1+p_2-1)} = e^{1\cdots(p_1-1)} \wedge e^{(p_1+1)\cdots(p_1+p_2-1)}. \tag{A.12}$$

$$\theta([XY]) = \theta(X) \centerdot \theta(Y). \tag{A.13}$$

$$-\frac{1}{2}\mathrm{tr}(XY) = \theta(X) \cdot \theta(Y). \tag{A.14}$$

$$\beta \centerdot \alpha = (-1)^{|\alpha||\beta|+1} \alpha \centerdot \beta, \tag{A.15}$$

$$(-1)^{|\alpha||\gamma|} \alpha \centerdot (\beta \centerdot \gamma) + \text{c.p.} = 0 = (-1)^{|\alpha||\gamma|} (\alpha \centerdot \beta) \centerdot \gamma + \text{c.p}, \tag{A.16}$$

where c.p. *denotes cyclic permutations of α, β, γ.*

If $|\alpha| = |\beta| = |\gamma| = 2$, then

$$\alpha \cdot (\beta \centerdot \gamma) = (\alpha \centerdot \beta) \cdot \gamma. \tag{A.17}$$

Note that (A.15) and (A.16) together state that $\centerdot$ is a graded Lie bracket.

Proof. Equation (A.12) follows from (A.11) since $(-1)^{p_1-1} a_{p_1}(e^{1\cdots p_1}) = e^{1\cdots(p_1-1)}$.

For (A.13), we compute

$$\theta([X, Y]) = \frac{1}{2}\left(X_{ij} Y_{jk} - Y_{ij} X_{jk}\right) e^{ik} = X_{ij} Y_{jk} e^{ik} = \frac{1}{4} X_{ij} Y_{k\ell} (e^{ij} \centerdot e^{k\ell}) = \theta(X) \centerdot \theta(Y).$$

To verify (A.14),

$$-\frac{1}{2}\mathrm{tr}(XY) = -\frac{1}{2} X_{ij} Y_{ji} = \frac{1}{4} X_{ij} Y_{kl} (e^{ij} \cdot e^{kl}) = \theta(X) \cdot \theta(Y).$$

For (A.15),

$$\beta \centerdot \alpha = (-1)^{|\beta|-1} (a_i \beta)(a_i \alpha) = (-1)^{|\alpha||\beta|+|\alpha|} (a_i \alpha)(a_i \beta) = (-1)^{|\alpha||\beta|+1} \alpha \centerdot \beta.$$

For (A.16), we compute

$$
\begin{aligned}
&(-1)^{|\alpha||\gamma|}\alpha \mathbin{\boldsymbol{.}} (\beta \mathbin{\boldsymbol{.}} \gamma) + \text{c.p.} \\
&= (-1)^{|\alpha|(|\gamma|+1)+|\beta|}(a_i\alpha)a_i((a_j\beta)(a_j\gamma)) + \text{c.p.} && \text{by (A.11)} \\
&= (-1)^{|\alpha|(|\gamma|+1)+|\beta|}(a_i\alpha)\left((a_{ij}\beta)(a_j\gamma) + (-1)^{|\beta|-1}(a_j\beta)(a_{ij}\gamma)\right) + \text{c.p.} && \text{by (A.7)} \\
&= (-1)^{|\alpha|(|\gamma|+1)+|\beta|}\Big((a_i\alpha)(a_{ij}\beta)(a_j\gamma)+ \\
&\qquad + (-1)^{|\beta|-1+(|\alpha|-1)(|\beta|-1+|\gamma|-2)}(a_j\beta)(a_{ij}\gamma)(a_i\alpha)\Big) + \text{c.p.} \\
&= (-1)^{|\alpha|(|\gamma|+1)+|\beta|}(a_j\alpha)(a_{ji}\beta)(a_i\gamma) + (-1)^{|\beta|(|\alpha|+1)+|\gamma|}(a_j\beta)(a_{ij}\gamma)(a_i\alpha) + \text{c.p.} \\
&= (-1)^{|\beta|(|\alpha|+1)+|\gamma|}(a_j\beta)(a_{ji}\gamma)(a_i\alpha) - (-1)^{|\beta|(|\alpha|+1)+|\gamma|}(a_j\beta)(a_{ji}\gamma)(a_i\alpha) + \text{c.p.} \\
&= 0.
\end{aligned}
$$

To verify (A.17) for $\alpha, \beta, \gamma \in \Lambda^2 V^*$, we compute

$$
\alpha \cdot (\beta \mathbin{\boldsymbol{.}} \gamma) = -\frac{1}{2}\operatorname{tr}\left(\theta^{-1}(\alpha)\left[\theta^{-1}(\beta)\theta^{-1}(\gamma)\right]\right) = -\frac{1}{2}\operatorname{tr}\left(\left[\theta^{-1}(\alpha)\theta^{-1}(\beta)\right]\theta^{-1}(\gamma)\right) = (\alpha \mathbin{\boldsymbol{.}} \beta)\cdot\gamma.
$$

□

A.1.7 Products on $\mathfrak{g} \otimes \Lambda^\bullet V^*$

Let G be a Lie group with corresponding Lie algebra $\mathfrak{g}$. Suppose further that $\mathfrak{g}$ is equipped with an invariant metric. Since infix notation (e.g. $\wedge, \cdot, \mathbin{\boldsymbol{.}}$) will be reserved for operations on forms, we will use outfix products for $\mathfrak{g}$. We denote the Lie bracket by $[\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, and the invariant metric by $\langle\ \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$.

Of primary interest is the case $\mathfrak{g} \cong \mathfrak{sp}(1) \cong \mathfrak{su}(2) \cong \mathfrak{so}(3)$. We fix Lie algebra isomorphisms

$$
\begin{aligned}
i &\mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}, \\
j &\mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}, \\
k &\mapsto \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \mapsto \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

We use the metrics

$$
\begin{aligned}
\langle \chi\xi \rangle &= \operatorname{Re}(\overline{\chi}\xi) \text{ for } \mathfrak{sp}(1), \\
\langle \chi\xi \rangle &= -\tfrac{1}{2}\operatorname{Tr}(\chi\xi) \text{ for } \mathfrak{u}(2), \\
\langle \chi\xi \rangle &= -\tfrac{1}{8}\operatorname{Tr}(\chi\xi) \text{ for } \mathfrak{so}(3).
\end{aligned}
\tag{A.18}
$$

Fixing a normalization on any one of these Lie algebras, the other normalizations are determined by the constraint that the Lie algebra isomorphisms are isometries.

Invariance of the metric means that

$$\langle [\xi\chi]\,\zeta\rangle = \langle \xi\,[\chi\zeta]\rangle \text{ for all } \chi, \xi, \zeta \in \mathfrak{g}. \tag{A.19}$$

Our products on $\mathfrak{g} \otimes \Lambda^\bullet V^*$ will be determined by specifying both an outfix product on $\mathfrak{g}$, and an infix product on $\Lambda^\bullet V^*$.

Proposition A.1.7. *For all $\alpha, \beta, \gamma \in \mathfrak{g} \otimes \Lambda^\bullet V^*$ of homogeneous degree, and $X \in \mathfrak{g} = \mathfrak{g} \otimes \Lambda^0 V^*$, the following identities hold:*

$$[\alpha \centerdot \beta] = \alpha \centerdot \beta + (-1)^{|\alpha||\beta|}\beta \centerdot \alpha \in \mathcal{U}(\mathfrak{g}) \otimes \Lambda^{|\alpha|+|\beta|-2} V^*. \tag{A.20}$$

$$[\beta \centerdot \alpha] = (-1)^{|\alpha||\beta|}\,[\alpha \centerdot \beta]. \tag{A.21}$$

$$0 = (-1)^{|\alpha||\gamma|}\,\langle [\alpha \centerdot \beta] \centerdot \gamma\rangle + \text{c.p.} \tag{A.22}$$

$$\langle \alpha \cdot [\beta X]\rangle = \langle [\alpha \cdot \beta]\, X\rangle \tag{A.23}$$

If $|\alpha| = |\beta| = |\gamma| = 2$, then

$$0 = [[\alpha \centerdot \beta] \cdot \gamma] + \text{c.p.} \tag{A.24}$$

$$\langle [\alpha \centerdot \beta] \cdot \gamma\rangle = \langle [\beta \centerdot \gamma] \cdot \alpha\rangle \tag{A.25}$$

Proof. By multilinearity, it suffices to prove the above identities for decomposable α, β, γ. Write $\alpha = \alpha_1 \otimes \alpha_2$, with $\alpha_1 \in \mathfrak{g}$ and $\alpha_2 \in \Lambda^\bullet V^*$, and similarly for β and γ.

For (A.20), we have the identity $[\alpha_1\beta_1] = \alpha_1\beta_1 - \beta_1\alpha_1$ in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, so

$$[\alpha \centerdot \beta] = [\alpha_1\beta_1] \otimes \alpha_2 \centerdot \beta_2 = \alpha_1\beta_1 \otimes \alpha_2 \centerdot \beta_2 + (-1)^{|\alpha||\beta|}\beta_1\alpha_1 \otimes \beta_2 \centerdot \alpha_2 = \alpha \centerdot \beta + (-1)^{|\alpha||\beta|}\beta \centerdot \alpha.$$

Equation (A.21) is a direct consequence of (A.20).

For (A.22) we compute

$$\begin{aligned}
&(-1)^{|\alpha||\gamma|}\,\langle [\alpha \centerdot \beta] \centerdot \gamma\rangle + \text{c.p.} \\
=&(-1)^{|\alpha||\gamma|}\,\langle [\alpha_1\beta_1]\,\gamma_1\rangle \otimes (\alpha_2 \centerdot \beta_2) \centerdot \gamma_2 + \text{c.p.} \\
=&\langle [\alpha_1\beta_1]\,\gamma_1\rangle \otimes \big((-1)^{|\alpha||\gamma|}(\alpha_2 \centerdot \beta_2) \centerdot \gamma_2 + \text{c.p.}\big) && \text{by (A.19)}, \\
=&0 && \text{by (A.16)}.
\end{aligned}$$

For (A.23) we compute

$$\langle \alpha \cdot [\beta X]\rangle = \langle \alpha_1\,[\beta_1 X]\rangle \otimes \alpha_2 \cdot \beta_2 = \langle [\alpha_1\beta_1]\, X\rangle \otimes \alpha_2 \cdot \beta_2 = \langle [\alpha \cdot \beta]\, X\rangle\,.$$

For (A.24) we compute

$$\begin{aligned}
&[[\alpha \centerdot \beta] \cdot \gamma] + \text{c.p.} \\
=&[[\alpha_1\beta_1]\,\gamma_1] \otimes (\alpha_2 \centerdot \beta_2) \cdot \gamma_2 + \text{c.p.} \\
=&([[\alpha_1\beta_1]\,\gamma_1] + \text{c.p.}) \otimes (\alpha_2 \centerdot \beta_2) \cdot \gamma_2 && \text{by (A.17)}, \\
=&0.
\end{aligned}$$

For (A.25), we combine (A.17) and (A.19) to get

$$\langle [\alpha \centerdot \beta] \cdot \gamma \rangle = \langle [\alpha_1\beta_1]\,\gamma_1 \rangle \otimes (\alpha_2 \centerdot \beta_2) \cdot \gamma_2 = \langle [\beta_1\gamma_1]\,\alpha_1 \rangle \otimes (\beta_2 \centerdot \gamma_2) \cdot \alpha_2 = \langle [\beta \centerdot \gamma] \cdot \alpha \rangle .$$

□

A.2 Geometry in a local frame

A.2.1 Lie and exterior derivatives

Let X be a n-manifold with boundary, equipped with a frame $\{e_i\}_{i=1}^n$ and corresponding dual coframe $\{e^i\}$. (If our manifold of interest has no global frame, then we restrict locally to a n-submanifold which does.)

Corresponding to the vector fields $\{e_i\}$, we get Lie derivatives $\{\mathcal{L}_i\}$ which act on functions $f \in \Omega^0(X)$. Since we allow for non-coordinate general frames, the functions $\mathcal{L}_1\mathcal{L}_2 f$ and $\mathcal{L}_2\mathcal{L}_1 f$ can differ.)

Suppose for the moment that X has a coordinate frame $\{e_\alpha\}$ arising from coordinates x^α. Then there is some change of frame $G \in \mathrm{GL}(TX)$ expressing our frame $\{e_i\}$ in terms of the coordinate frame $\{e_\alpha\}$ by $e_i = G^\alpha{}_i e_\alpha$. Thus

$$\mathcal{L}_i = G^\alpha{}_i \frac{\partial}{\partial x^\alpha}.$$

We define functions

$$c_i{}^k{}_j := (G^{-1})^k{}_\alpha (\mathcal{L}_i G^\alpha{}_j - \mathcal{L}_j G^\alpha{}_i)$$

which are independent of the choice of coordinate frame, and for all $f \in \Omega^0(X)$ satisfy

$$\mathcal{L}_i\mathcal{L}_j f - \mathcal{L}_j\mathcal{L}_i f = c_i{}^k{}_j \mathcal{L}_k f, \qquad c_i{}^k{}_j = -c_j{}^k{}_i, \qquad \begin{aligned} 0 = {} & c_i{}^m{}_j c_m{}^k{}_\ell - \mathcal{L}_i c_j{}^k{}_\ell \\ & + c_j{}^m{}_\ell c_m{}^k{}_i - \mathcal{L}_j c_\ell{}^k{}_i \\ & + c_\ell{}^m{}_i c_m{}^k{}_j - \mathcal{L}_\ell c_i{}^k{}_j. \end{aligned}$$

In the general case when there is no coordinate frame for X, there are still functions $c_i{}^k{}_j$ associated with our frame, which satisfy these relations.

Let $\Omega^\bullet(X)$ denote the space of smooth sections of $\Lambda^\bullet(T^*X)$. It comes equipped with a natural action of the oscillator algebra $\Theta^\bullet(TX)$. Given a form $\omega = \omega_I e^I$, we define the *function-only Lie derivatives* by

$$\mathcal{L}^0_i(\omega) := (\mathcal{L}_i\omega_I)e^I.$$

These derivatives define operators $\underline{\mathcal{L}}^0_i$ on $\Omega^\bullet(T^*X)$. Thus we have an operator algebra generated by $\left\{\iota_i, \epsilon^i, \underline{\mathcal{L}}^0_i\right\}$ in degrees $(-1, 1, 0)$, and functions $\left\{c_i{}^k{}_j\right\}$ obeying the following graded-commutation relations:

$$\left[\epsilon^i, \epsilon^j\right] = 0 = \left[\iota_i, \iota_j\right], \quad \left[\epsilon^i, \iota_j\right] = \delta^i_j, \quad \left[\underline{\mathcal{L}}^0_i, f\right] = \mathcal{L}_i f, \quad \left[\underline{\mathcal{L}}^0_i, \underline{\mathcal{L}}^0_j\right] = c_i{}^k{}_j \underline{\mathcal{L}}^0_k, \quad \left[\underline{\mathcal{L}}^0_i, \iota_j\right] = 0 = \left[\underline{\mathcal{L}}^0_i, \epsilon^j\right].$$

Given a local frame, If G is a change of frame, then G naturally acts as an automorphism of this operator algebra:

$$G(\iota_i) = G^{i'}{}_i \iota_{i'}, \qquad G(\epsilon^i) = (G^{-1})^i{}_{i'}\epsilon^{i'}, \qquad G(\underline{\mathcal{L}}^0_i) = G^{i'}{}_i \left(\underline{\mathcal{L}}^0_{i'} + (G^{-1})^k{}_{k'}\left(\mathcal{L}_{i'}G^{k'}{}_j\right) G(\epsilon^j a_k)\right),$$

$$G(c)_i{}^k{}_j = G^{i'}{}_i (G^{-1})^k{}_{k'} G^{j'}{}_j \left(c_{i'}{}^{k'}{}_{j'} + (G^{-1})^{j''}{}_{j'}(\mathcal{L}_{i'}G^{k'}{}_{j''}) - (G^{-1})^{i''}{}_{i'}(\mathcal{L}_{j'}G^{k'}{}_{i''})\right),$$

so the operators $\left\{G(\iota_i), G(\epsilon^i), G(\underline{\mathcal{L}}^0_i)\right\}$ and functions $\left\{G(c)_i{}^k{}_j\right\}$ satisfy the same relations.

We define the Lie derivatives and the exterior derivative respectively as

$$\underline{\mathcal{L}}_i := \underline{\mathcal{L}}^0_i - c_i{}^k{}_j \epsilon^j \iota_k, \qquad\qquad \underline{d} := \epsilon^i \underline{\mathcal{L}}_i + \frac{1}{2} c_i{}^k{}_j \epsilon^{ij} \iota_k.$$

They satisfy

$$[\underline{\mathcal{L}}_i, f] = \mathcal{L}_i f, \quad \left[\underline{\mathcal{L}}_i, \underline{\mathcal{L}}_j\right] = c_i{}^k{}_j \underline{\mathcal{L}}_k + (\mathcal{L}_k c_i{}^\ell{}_j)\epsilon^k \iota_\ell, \quad \left[\underline{\mathcal{L}}_i, \iota_j\right] = c_i{}^k{}_j \iota_k, \quad \left[\underline{\mathcal{L}}_i, \epsilon^k\right] = -c_i{}^k{}_j \epsilon^j,$$

$$[\underline{d}, \underline{d}] = 2\underline{d}^2 = 0, \qquad [\underline{d}, \underline{\mathcal{L}}_i] = 0, \qquad [\underline{d}, \iota_i] = \underline{\mathcal{L}}_i, \qquad \left[\underline{d}, \epsilon^k\right] = -\tfrac{1}{2} c_i{}^k{}_j \epsilon^{ij}, \qquad G(\underline{d}) = \underline{d}.$$

Again, if G is a change of frame, then $\left\{G(\iota_i), G(\epsilon^i), G(\underline{\mathcal{L}}^0_i), G(\underline{\mathcal{L}}_i), \underline{d}, G(c)_i{}^k{}_j\right\}$ satisfy the same relations. Note how the $\{\underline{\mathcal{L}}_i\}$ can be reconstructed from $\underline{d}$ via $[\underline{d}, \iota_i] = \underline{\mathcal{L}}_i$.

A.2.2 The Levi-Civita connection

Given a manifold with metric tensor $g = \left\{g_{ij}\right\}$, we define the Christoffel symbols

$$\Gamma_i{}^k{}_j := \frac{1}{2} g^{kk'} \left(\mathcal{L}_i g_{k'j} + \mathcal{L}_j g_{k'i} - \mathcal{L}_{k'} g_{ij} + c_{k'}{}^{i'}{}_j g_{ii'} + c_{k'}{}^{j'}{}_i g_{jj'} + c_i{}^{k''}{}_j g_{k'k''}\right),$$

or equivalently

$$\Gamma_{ikj} := \frac{1}{2}\left(\mathcal{L}_i g_{kj} + \mathcal{L}_j g_{ki} - \mathcal{L}_k g_{ij} + c_{kij} + c_{kji} + c_{ikj}\right).$$

They satisfy

$$\Gamma_i{}^k{}_j - \Gamma_j{}^k{}_i = c_i{}^k{}_j, \tag{A.26}$$

$$\Gamma_{irs} + \Gamma_{isr} = \mathcal{L}_i g_{rs},$$
$$\Gamma_i{}^{rs} + \Gamma_i{}^{sr} = -\mathcal{L}_i g^{rs}, \tag{A.27}$$

and transform as

$$G(\Gamma)_i{}^k{}_j = G^{i'}{}_i (G^{-1})^k{}_{k'} G^{j'}{}_j \left(\Gamma_{i'}{}^{k'}{}_{j'} + (G^{-1})^{j''}{}_{j'} \mathcal{L}_{i'} G^{k'}{}_{j''} \right),$$
$$= G^{i'}{}_i (G^{-1})^k{}_{k'} G^{j'}{}_j \left(\Gamma_{i'}{}^{k'}{}_{j'} - G^{k'}{}_{k''} \mathcal{L}_{i'} (G^{-1})^{k''}{}_j \right).$$

Define the Levi-Civita derivative ∇T of a tensor such as $T = \left\{ T^i{}_j{}^k{}_\ell \right\}$ to be

$$(\nabla T)_h{}^i{}_j{}^k{}_\ell := \mathcal{L}_h T^i{}_j{}^k{}_\ell + \Gamma_h{}^i{}_{i'} T^{i'}{}_j{}^k{}_\ell - \Gamma_h{}^{j'}{}_j T^i{}_{j'}{}^k{}_\ell + \Gamma_h{}^k{}_{k'} T^i{}_j{}^{k'}{}_\ell - \Gamma_h{}^{\ell'}{}_\ell T^i{}_j{}^k{}_{\ell'},$$

where Γ_h acts on upper indices v^i as $\Gamma_h{}^i{}_{i'} v^{i'}$, and on lower indices α_i as $-\Gamma_h{}^{i'}{}_i \alpha_{i'}$. These respective actions of Γ_h are the standard representation of $\mathfrak{gl}(TX)$ on $\Gamma(TX)$, and the dual representation (A.2) of $\mathfrak{gl}(TX)$ on $\Gamma(T^*X)$.

The metric compatibility relations (A.27) are equivalent to the statement that raising/lowering commute with the operation of taking a covariant derivative.

On differential forms, the covariant derivative is defined by

$$\nabla_i \omega := \mathcal{L}_i^0(\omega) - \Gamma_i{}^k{}_j \epsilon^j \iota_k \omega,$$

and the corresponding operator satisfies

$$\left[\underline{\nabla}_i, \iota_j \right] = \iota_k \Gamma_i{}^k{}_j, \qquad \left[\underline{\nabla}_i, \epsilon^j \right] = -\epsilon^k \Gamma_i{}^j{}_k, \qquad \left[\underline{\nabla}_i, \epsilon_j \right] = \epsilon_k \Gamma_i{}^k{}_j, \qquad \left[\underline{\nabla}_i, \iota^k \right] = -\iota^j \Gamma_i{}^k{}_j,$$

$$\underline{d} = \epsilon^i \underline{\nabla}_i, \qquad \underline{d}^* = -\iota_i \underline{\nabla}_i, \qquad \left[\underline{\nabla}_i, \underline{\nabla}_j \right] = c_i{}^k{}_j \underline{\nabla}_k + R_{ijk\ell} \epsilon^k \iota_\ell,$$

where R is the curvature tensor, yet to be defined in (A.28).

The covariant derivative of a multivector v is given by $\nabla_i v := (\mathcal{L}_i^0 + \Gamma_i{}^k{}_j \iota_k \epsilon^j) v$. The covariant derivative then satisfies the compatibility relations

$$\mathcal{L}_i(\alpha \cdot v) = (\nabla_i \alpha) \cdot v + \alpha \cdot \nabla_i v,$$
$$\mathcal{L}_i(\alpha \cdot \beta) = (\nabla_i \alpha) \cdot \beta + \alpha \cdot \nabla_i \beta,$$
$$\mathcal{L}_i(v \cdot w) = (\nabla_i v) \cdot w + v \cdot \nabla_i w.$$

We define the Riemann curvature tensor $R_{ij}{}^k{}_\ell$ by

$$R_{ij}{}^k{}_\ell : = \mathcal{L}_i \Gamma_j{}^k{}_\ell - \mathcal{L}_j \Gamma_i{}^k{}_\ell + \Gamma_i{}^k{}_m \Gamma_j{}^m{}_\ell - \Gamma_j{}^k{}_m \Gamma_i{}^m{}_\ell - c_i{}^m{}_j \Gamma_m{}^k{}_\ell. \tag{A.28}$$

Note that for each i, j, $R_{ij}{}^\bullet{}_\bullet \in \mathfrak{o}(TM)$. Furthermore, R satisfies

$$R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij}, \qquad R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0. \tag{A.29}$$

On a sphere with a standard metric,

$$R_{ijij} > 0 \text{ for } i \neq j.$$

The second Bianchi identity is

$$(\nabla R)_{hij}{}^{k}{}_{\ell} + (\nabla R)_{ijh}{}^{k}{}_{\ell} + (\nabla R)_{jhi}{}^{k}{}_{\ell} = 0.$$

If g is positive-definite and X comes equipped with an orientation, then define $\sqrt{g} := \pm\sqrt{\det g}$, with the sign depending on the orientation of the frame so that the volume form

$$d\mathrm{vol}_X := \sqrt{g} e^{1\cdots n}$$

is positive. It transforms as $G(d\mathrm{vol}_X) = d\mathrm{vol}_X$.

We define the divergence of a vector field v by

$$\mathrm{div}(v) := e^i \cdot \nabla_i v = \mathcal{L}_j v^j + \Gamma_j{}^j{}_i v^i \in \Omega^0(X),$$

which satisfies

$$\mathrm{div}(e_i) = \Gamma_j{}^j{}_i = \mathcal{L}_i \ln\sqrt{g} + c_j{}^j{}_i.$$

The Lie derivative of $d\mathrm{vol}_X$ is

$$\mathcal{L}_i \, d\mathrm{vol}_X = \mathrm{div}(e_i)\, d\mathrm{vol}_X.$$

A.2.3 Integration by parts

For oriented X, the volume form allows us to define the Hodge star and integration of functions, leading to integration by parts.

The Hodge star $\star$ is characterized by the relation $\alpha \wedge \star\beta = (\alpha \cdot \beta)\, d\mathrm{vol}_M$. It satisfies the properties

$$\star^2 = (-1)^{k(n-k)}, \qquad \epsilon^i \star (-1)^{k+1} = \star\iota^i, \qquad \iota_i \star (-1)^k = \star\epsilon_i,$$

$$d^* = \star d \star (-1)^{n(k+1)+1}, \qquad \star L = (\mathrm{tr}(L) - L^*)\star, \qquad \star e^I = \frac{\sqrt{g}}{|J|!} \varepsilon_{IJ}^{1\cdots n} e^J,$$

where $n = \dim X$, k is the operator which gives the degree of a form, ϵ and ι are wedge and contraction operators of A.1.4, $L \in \mathrm{End}(TX)$ acts as (A.9), and ε is the antisymmetric tensor which gives the relative sign of two permutations.

Integration of a function is given by

$$\int_X f := \int_X f \, d\mathrm{vol}_X,$$
$$\|f\| := \sqrt{\int_M f^2 d\mathrm{vol}_M},$$
$$\|f\|_{L^p_k} := \left(\int_M \sum_{0 \le |I| \le k} |\nabla_I f|^p \, d\mathrm{vol}_M \right)^{1/p}.$$

The pointwise metric pairing on forms extends to the global metric pairing $\int_X \alpha \cdot \beta$. If α and β are vector bundle-valued differential forms, and the bundle has metric $\langle\rangle$, then the pairing is $\int_X \langle \alpha \cdot \beta \rangle$.

Given a differential operator D, the *formal metric adjoint* D^* is the differential operator such that for compactly supported α and β,

$$\int_X \langle D\alpha \cdot \beta \rangle = \int_X \langle \alpha \cdot D^* \beta \rangle .$$

As noted in Remark A.1.1, we use the notation D^* for the metric adjoint although it conflicts with the notation for the dual map.

We use the *outward normal first* convention to induce an orientation on ∂X so that $\int_{\partial X} \beta = \int_X d\beta$ for $\beta \in \Omega^{n-1}(X)$. This gives us an identity for a boundary integral:

$$\int_{\partial X} f \, \iota_i d\mathrm{vol}_X = \int_X d(\iota_i f \, d\mathrm{vol}_X) = \int_X \mathcal{L}_i(f \, d\mathrm{vol}_X) = \int_X \left(\mathcal{L}_i f + f \, \mathrm{div}(e_i)\right) \, d\mathrm{vol}_X. \tag{A.30}$$

Note that $\mathcal{L}_i(\alpha \cdot \beta) = (\nabla_i \alpha) \cdot \beta + \alpha \cdot (\nabla_i \beta)$. Substituting $f = \alpha \cdot \beta$ into (A.30), we get our integration by parts formula

$$\int_X (\nabla_i \alpha) \cdot \beta \, d\mathrm{vol}_X = \int_{\partial X} (\alpha \cdot \beta) \, \iota_i d\mathrm{vol}_X + \int_M \alpha \cdot (\nabla_i^* \beta) \, d\mathrm{vol}_X,$$

where the formal metric adjoint of ∇ is

$$\nabla_i^* = -\left(\nabla_i + \mathrm{div}(e_i)\right). \tag{A.31}$$

We extend the adjoint of ∇_i to bundle coefficients in (A.36).

For the Weitzenböck formula, we will need the covariant Laplacian denoted $\nabla^* \nabla$, where ∇ is given by $e^i \otimes \nabla_i \in T^*M \otimes \Omega^\bullet(M)$. We compute

$$\begin{aligned} &\int_X \langle \nabla \alpha \cdot \nabla \beta \rangle \\ &= \int_X \left\langle \nabla_i \alpha \cdot g^{ij} \nabla_j \beta \right\rangle \\ &= \int_{\partial X} \alpha \cdot (g^{ij} \nabla_j \beta) \, \iota_i d\mathrm{vol}_X + \int_X \langle \alpha \cdot \nabla^* \nabla \beta \rangle , \end{aligned} \tag{A.32}$$

where $\nabla^* \nabla = \nabla_i{}^* g^{ij} \nabla_j$.

A.3 Principal bundles

A.3.1 Fiber bundles, Čech cochains, and associated bundles

In this section, we introduce the formalism of non-abelian Čech cohomology. This is not meant to be a rigorous treatment, but rather a roadmap of some general principles (and principals).

Let $\pi : E \to X$ be a fiber bundle with fiber F_0 and structure group $G \subset \mathrm{Diff}(F_0)$. Given an open cover $\{U_\alpha\}$ of X, and a collection of smooth local trivializations $\Phi = \{\phi_\alpha : U_\alpha \times F_0 \to E\}$, we obtain a cocycle

$$\begin{aligned} \tau_{\alpha\beta} : U_\alpha \cap U_\beta &\longrightarrow G, \\ \tau_{\alpha\beta} &= \phi_\alpha^{-1}\phi_\beta. \end{aligned}$$

The cocycle condition is $\tau_{\alpha\beta}\tau_{\beta\gamma} = \tau_{\alpha\gamma}$.

Given any cocycle τ and a representation $\rho : G \to \mathrm{Diff}(F)$, we construct the associated bundle, which we denote

$$P_\tau \times_\rho F := \frac{\coprod_\alpha U_\alpha \times F}{\llbracket x_\beta, f_\beta \rrbracket \sim \llbracket x_\alpha, f_\alpha \rrbracket}, \text{ where } f_\alpha = \rho(\tau_{\alpha\beta}) f_\beta \text{ for all } \alpha, \beta.$$

(We have not yet defined the principal bundle P_τ, so for the moment this is only notation.) Implicit in τ are the canonical trivializations

$$\begin{aligned} U_\alpha \times F &\longrightarrow \tau \times_\rho F, \\ (x_\alpha, f_\alpha) &\mapsto \llbracket x_\alpha, f_\alpha \rrbracket. \end{aligned}$$

Furthermore, cocycle associated to $P_\tau \times_\rho F$ with the canonical trivializations is $\rho(\tau)$.

We now consider some special cases. In the case $F = F_0$ and $\rho = \rho_0$ is the original representation of G on F_0, then the original trivializations $\Phi = \{\phi_\alpha\}$ glue to give an isomorphism

$$\Phi : P_\tau \times_{\rho_0} F_0 \longrightarrow E.$$

The moral of this example is that a fiber bundle E is determined by its associated cochain τ via the construction $P_\tau \times_{\rho_0} F_0$, and a canonical isomorphism $E \cong P_\tau \times_{\rho_0} F_0$ is determined by the local trivializations Φ.

In the case $F = G$ and $\rho = m_L$ is left-multiplication on the fiber G, we get the principal bundle

$$P_\tau := P_\tau \times_{m_L} G.$$

Since m_L is a faithful representation, the Čech cocycle of P_τ is τ itself. A local section $\psi : U \to P_\tau$ represented over U_β by $\llbracket x_\beta, \psi_\beta \rrbracket$ corresponds to a local trivialization of $\tau \times_\rho F$ for *any* F and ρ. This local trivialization is given over $U_\alpha \cap U_\beta$ by

$$\begin{aligned} \rho(\psi) : (U_\alpha \cap U_\beta) \times F &\longrightarrow P_\tau \times_\rho F \\ (x_\beta, f) &\mapsto \llbracket x_\beta, \rho(\psi_\beta) f \rrbracket \sim \llbracket x_\alpha, \rho(\tau_{\alpha\beta})\rho(\psi_\beta) f \rrbracket = \llbracket x_\alpha, \psi_\alpha f \rrbracket. \end{aligned}$$

The moral of this example is that the principal bundle P_τ encodes the cocycle τ.

Note that P_τ has a natural right G-action given by $[\![x_\alpha, \psi_\alpha]\!]\,\mathrm{g} := [\![x_\alpha, \psi_\alpha \mathrm{g}]\!]$. For a general principal bundle P, we define

$$P \times_\rho F := \frac{P \times F}{[\![[\![x_\alpha, \psi_\alpha]\!], f]\!] \sim [\![[\![x_\alpha, \psi_\alpha]\!]\,\mathrm{g}^{-1}, \rho(\mathrm{g}) f]\!]}.$$

For $P = P_\tau$, the resulting space is equivalent to the old $P_\tau \times_\rho F$ via the diffeomorphism $[\![[\![x_\alpha, \psi_\alpha]\!], f]\!] \mapsto [\![x_\alpha, \rho(\psi_\alpha) f]\!]$.

Now consider the case $F = G$ and $\rho = \mathrm{Ad}$ is the adjoint representation. Then a section $\mathrm{g} : X \to P_\tau \times_{\mathrm{Ad}} G$ corresponds to an automorphism $\rho(\mathrm{g})$ of $P_\tau \times_\rho F$ for *any* F and ρ. This automorphism is given over $U_\alpha \cap U_\beta$ by

$$\begin{aligned} \rho(\mathrm{g}) : P_\tau \times_\rho F &\longrightarrow P_\tau \times_\rho F \\ [\![x_\beta, f_\beta]\!] &\mapsto [\![x_\beta, \rho(\mathrm{g}_\beta) f_\beta]\!] \sim [\![x_\alpha, \rho(\tau_{\alpha\beta}\mathrm{g}_\beta)(\tau_{\alpha\beta}^{-1} f_\alpha)]\!] = [\![x_\alpha, \rho(\mathrm{g}_\alpha) f_\alpha]\!]. \end{aligned}$$

This situation can be summarized nicely with the language of non-abelian Čech cohomology. Recall that in abelian cohomology, we have an exact sequence

$$0 \to Z^0 \to C^0 \to Z^1 \to H^1 \to 0.$$

for a cochain complex $(C^\bullet, \delta)$ with cocycles $Z^\bullet$ and cohomology $H^\bullet$. This sequence extends to the non-abelian context in the following way:

$$\begin{tikzcd}[column sep=small]
 & & \overbrace{\check{C}^0(P)}^{\substack{\text{trivialization } \Phi=\{\phi_\alpha\}, \\ \phi_\alpha : U_\alpha \to P, \\ \mathrm{g}\Phi = \{\phi_\alpha \mathrm{g}_\alpha^{-1}\}}} \arrow[d, squiggly] \arrow[dr, "\delta : \Phi \mapsto \{\tau_{\alpha\beta} = \phi_\alpha^{-1}\phi_\beta\}"] & & & \\
0 \arrow[r] & \underbrace{\check{Z}^0(P \times_{\mathrm{Ad}} \mathbf{G})}_{\substack{\text{bundle automorphisms } \mathrm{g}=\{\mathrm{g}_\alpha\}, \\ \mathrm{g}_\alpha = \tau_{\alpha\beta}\mathrm{g}_\beta\tau_{\alpha\beta}^{-1}}} \arrow[r] & \underbrace{\check{C}^0(\mathbf{G})}_{\substack{\text{change of} \\ \text{trivialization} \\ \mathrm{g}=\{\mathrm{g}_\alpha\}}} \arrow[r, squiggly] & \underbrace{\check{Z}^1(\mathbf{G})}_{\substack{\text{gluing data} \\ \tau_{\alpha\beta}\tau_{\beta\gamma} = \tau_{\alpha\gamma} \\ \mathrm{g}\tau = \{\mathrm{g}_\alpha \tau_{\alpha\beta} \mathrm{g}_\beta^{-1}\}}} \arrow[r] & \underbrace{\check{H}^1(\mathbf{G})}_{\substack{\text{isomorphism} \\ \text{classes (based at } [\![P]\!])}} \arrow[r] & 0.
\end{tikzcd}$$

We now explain the notation. The base space X and open cover $\{U_i\}$ are implicit throughout. For simiplicitly, we ignore issues of refinements of open covers. Instead of abelian groups, there are *set*-valued functors $\check{C}^0$, $\check{Z}^0$, $\check{Z}^1$ and $\check{H}^1$. Straight arrows denote maps of sets, while the squiggly arrows emanating from $\check{C}^0(\mathbf{G})$ denote a group action. The cohomology coefficient $\mathbf{G}$ denotes some subsheaf $\mathbf{G} \subset C^\infty(X; G)$. For example, if X is a complex manifold, then $\mathbf{G}$ might denote the sheaf of holomorphic maps from X to $\mathrm{GL}(\mathbb{C}^n)$. In this case, a principal $\mathbf{G}$-bundle will determine a holomorphic vector bundle of rank n.

For a fiber bundle E with fiber F, the notation $\check{C}^0(E)$ denotes the set of maps $\coprod_\alpha U_\alpha \to F$, while $\check{Z}^0(E)$ denotes the subset of such maps $\{s_\alpha\}$ which satisfy the cocycle condition $s_\alpha = \rho(\tau_{\alpha\beta}) s_\beta$. Thus $\check{Z}^0 = \check{H}^0$ corresponds to global sections. The set $\check{Z}^1(\mathbf{G})$ denotes the set of $\mathbf{G}$-maps $\tau : \coprod_{\alpha,\beta} U_\alpha \cap U_\beta \to G$

which satisfy the cocycle condition $\tau_{\alpha\beta}\tau_{\beta\gamma}\tau_{\alpha\gamma}^{-1} = 1$. The set $\check{H}^1(\mathbf{G})$ consists of the $\check{C}^0(\mathbf{G})$-orbits in $\check{Z}^1(\mathbf{G})$ under the action

$$g\tau = \left\{ g_\alpha \tau_{\alpha\beta} g_\beta^{-1} \right\}. \tag{A.33}$$

"Exactness" from right to left, amounts to the following facts:

- Every isomorphism class of principal $\mathbf{G}$-bundle admits some gluing data τ.
- P_τ is isomorphic to P if and only if it arises from some trivialization Φ of P.
- Two trivializations Φ_1 and Φ_2 determine the same τ if and only if they differ by a bundle automorphism.

This formalism is capable of reducing some otherwise complicated theorems to simple diagram chasing. For example, suppose we have a central extension of our structure group $0 \to \mathbf{Z} \to \tilde{\mathbf{G}} \to \mathbf{G} \to 0$. Then we get an "exact sequence"

$$\check{H}^1(\mathbf{Z}) \to \check{H}^1(\tilde{\mathbf{G}}) \to \check{H}^1(\mathbf{G}) \to \check{H}^2(\mathbf{Z}),$$

where the sets on each end are traditional abelian cohomology groups since Z is abelian. For example, we have the standard theorem

Theorem A.3.1. *A real oriented vector bundle V admits a* Spin^c*-structure if and only if $w_2(V) \in H^2(\mathbb{Z}_2)$ admits an integral lift.*

Proof. It suffices to show that $W_3 \in H^3(\mathbb{Z})$ is the obstruction to lifting an SO-structure, and that $W_3 = 0$ if and only if $w_2 \in H^2(\mathbb{Z})$ mod 2. The proof follows from chasing the following diagrams, where colinear arrows are exact, and the dotted arrows indicate boundary maps induced on cohomology.

$$
\begin{tikzcd}
 & 0 \arrow[dd] & & & & & & & \\
0 \arrow[dr] & & & 0 \arrow[dd] & & & & & \\
 & \mathbb{Z} \arrow[drr, "2"] \arrow[dd, "1"] & & & 0 \arrow[dd] & & 0 \arrow[dd] & 0 \arrow[dd] & \\
0 \arrow[dr] & & & \mathbb{Z} \arrow[dr, "\mathrm{mod}\ 2"] \arrow[dd, "1"] & & & & & \\
 & \mathbb{Z} \arrow[drr, "2"] \arrow[dd] & 0 \arrow[rr] & & \mathbb{Z}_2 \arrow[rr] \arrow[dr] \arrow[dd, "(-1)^{\bullet}"] \arrow[uulll, dotted, bend right, "\beta"'] & & \mathrm{Spin} \arrow[r] \arrow[dd] & \mathrm{SO} \arrow[r] \arrow[dd, "\mathrm{id}"] & 0 \\
 & & & \mathbb{R} \arrow[dr, "e^{\pi i \bullet}"] \arrow[dd, "e^{2\pi i \bullet}"] & & 0 & & & \\
 & 0 \arrow[drr] & 0 \arrow[rr] & & \mathrm{U}(1) \arrow[rr] \arrow[dr] \arrow[dd, "(\bullet)^2"] \arrow[uulll, dotted, bend right, "\cong"'] & & \mathrm{Spin}^c \arrow[r] \arrow[dd] & \mathrm{SO} \arrow[r] \arrow[dd] & 0 \\
 & & & \mathrm{U}(1) \arrow[dr, "\mathrm{id}"] \arrow[dd] & & 0 & & & \\
 & & 0 \arrow[rr] & & \mathrm{U}(1) \arrow[rr, "\mathrm{id}"] \arrow[dr] \arrow[dd] & & \mathrm{U}(1) \arrow[r] \arrow[dd] & 0 & \\
 & & & 0 & & 0 & & & \\
 & & & & 0 & & 0 & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & & \check{H}^2(\mathbb{Z}) \arrow[dr, "\mathrm{mod}\ 2"] & & \\
\check{H}^1(\mathbb{Z}_2) \arrow[rr] \arrow[dd, "(-1)^{\bullet}"] & & \check{H}^1(\mathrm{Spin}) \arrow[r] \arrow[dd] & \check{H}^1(\mathrm{SO}) \arrow[rr, "w_2"] \arrow[dd, "\mathrm{id}"] & & \check{H}^2(\mathbb{Z}_2) \arrow[dr, "\beta"] \arrow[dd, "(-1)^{\bullet}"] & \\
 & & & & & & \check{H}^3(\mathbb{Z}) \arrow[dd, "1"] \\
\check{H}^1(\mathrm{U}(1)) \arrow[rr] \arrow[dr, "\cong"] & & \check{H}^1(\mathrm{Spin}^c) \arrow[r] & \check{H}^1(\mathrm{SO}) \arrow[rr] \arrow[drrr, "W_3"'] & & \check{H}^2(\mathrm{U}(1)) \arrow[dr, "\cong"] & \\
 & \check{H}^2(\mathbb{Z}) & & & & & \check{H}^3(\mathbb{Z})
\end{tikzcd}
$$

□

Another application of this formalism is to understand reductions of the structure sheaf.

Suppose we have an exact sequence $0 \to \mathbf{H} \to \mathbf{G} \to \mathbf{G}/\mathbf{H} \to 0$, where $\mathbf{H}$ need not be normal. We get the following "exact sequence" sequence in Čech cohomology:

$$\underbrace{\check{H}^0(P \times_{\mathrm{Ad}} \mathbf{G})}_{\text{bundle automorphisms}} \rightsquigarrow \underbrace{\check{H}^0(P \times_{m_L} \mathbf{G}/\mathbf{H})}_{\text{reductions}} \longrightarrow \underbrace{\check{H}^1(\mathbf{H})}_{\substack{\text{isomorphism}\\ \text{classes of}\\ \mathbf{H}\text{-bundles}}} \longrightarrow \underbrace{\check{H}^1(\mathbf{G})}_{\substack{\text{isomorphism}\\ \text{classes of}\\ \mathbf{G}\text{-bundles}\\ \text{based at } [\![P]\!]}}. \tag{A.34}$$

Definition A.3.2. A *reduction* to a subsheaf $\mathbf{H} \subset \mathbf{G}$ is a section $R \in \check{Z}^0(P \times_{m_L} \mathbf{G}/\mathbf{H})$, where the representation m_L is the action induced by left multiplication.

If $P = P_\tau$ for some $\tau \in \check{Z}^1(\mathbf{G})$, and if R is a reduction, then R is represented by some $\tilde{R} = \{\tilde{R}_\alpha\} \in \check{C}^0(\mathbf{G})$ which obeys the compatibility condition $\tilde{R}_\alpha = \tau_{\alpha\beta}\tilde{R}_\beta\eta_{\alpha\beta}^{-1}$ for some $\eta \in \check{C}^1(\mathbf{H})$. This compatibility condition is equivalent to $\tau_{\alpha\beta} = \tilde{R}_\alpha\eta_{\alpha\beta}\tilde{R}_\beta$. Comparing with (A.33), we see that $\tilde{R}$ defines a change of trivialization which satisfies $\tau = \tilde{R}\eta$. Since $\eta = \tilde{R}^{-1}\tau$ is in the $\check{C}^0(\mathbf{G})$-orbit of $\tau \in \check{Z}^1(\mathbf{G})$, it follows that $\eta \in \check{Z}^1(\mathbf{G}) \cap \check{C}^1(\mathbf{H}) = \check{Z}^1(\mathbf{H})$. Any other representative $\tilde{R}'$ for R is related by $\tilde{R}'_\alpha = \tilde{R}_\alpha h_\alpha^{-1}$ for some $h \in \check{C}^0(\mathbf{H})$. It follows that the corresponding η' is $\{\eta'_{\alpha\beta}\} = \{h_\alpha\eta_{\alpha\beta}h_\beta^{-1}\}$, so $\eta' = h\eta$ belongs to the same $\check{C}^0(\mathbf{H})$-orbit, and thus $\eta \in \check{H}^1(H)$ is well-defined. In summary, the middle map of (A.34) is

$$\begin{aligned} \check{H}^0(P \times_{m_\tau^{-1}} \mathbf{G}/\mathbf{H}) &\longrightarrow \check{H}^1(\mathbf{H}), \\ R &\mapsto [\![\tilde{R}^{-1}\tau]\!] = [\![\{\tilde{R}_\alpha^{-1}\tau_{\alpha\beta}\tilde{R}_\beta\}]\!]. \end{aligned}$$

Finally, "exactness" means that

- The structure sheaf of P reduces from $\mathbf{G}$ to $\mathbf{H}$ if and only if there exists a reduction.
- Two reductions R_1 and R_2 lead to isomorphic $\mathbf{H}$-bundles if and only if R_1 and R_2 are related by a bundle automorphism.

Verifying these two points is a routine exercise in chasing the definitions.

For a trivial application of reduction when X is a point and $\mathbf{G} = \mathrm{GL}(\mathbb{C}^n)$, we prove:

Theorem A.3.3. *Let V be a vector space isomorphic to $\mathbb{C}^n$. Then a reduction from the general frame bundle $\mathrm{Fr}_{\mathrm{GL}}(V) = \mathrm{GL}(\mathbb{C}^n, V)$ to the unitary frame bundle $\mathrm{Fr}_{\mathrm{U}}(V) = \mathrm{U}(\mathbb{C}^n, V)$ is equivalent to a Hermitian metric $h \in \mathrm{Met}(V)$.*

Proof. There is an exact sequence

$$0 \longrightarrow \mathrm{U}(\mathbb{C}^n) \longrightarrow \mathrm{GL}(\mathbb{C}^n) \longrightarrow \mathrm{Met}(\mathbb{C}^n) \longrightarrow 0,$$

where the map $\mathrm{GL}(\mathbb{C}^n) \to \mathrm{Met}(\mathbb{C}^n)$ is $\mathrm{g} \mapsto (\mathrm{g}^{-1})^*\mathrm{g}^{-1}$. Associated to each frame $\phi : \mathbb{C}^n \to V$, we get the Hermitian metric $h = (\phi^{-1})^*\phi^{-1}$. It's clear that ϕ is orthonormal with respect to this metric. Conversely, given a Hermitian metric h, we can represent it as $(\phi^{-1})^*\phi^{-1}$ for some frame ϕ. □

A more interesting application is

Theorem A.3.4. *Over a complex manifold X, a reduction from a smooth vector bundle $E \to X$ to a holomorphic vector bundle is equivalent to a $\overline{\partial}$-operator on E which satisfies $\overline{\partial}^2 = 0$.*

There is a sequence of sheaves given by

$$0 \longrightarrow \mathcal{O}(\mathrm{GL}(n)) \longrightarrow C^\infty(\mathrm{GL}(n)) \longrightarrow \mathrm{Hol}(n) \longrightarrow 0.$$

The map to $\mathrm{Hol}(n)$ is given by $\mathrm{g} \mapsto \underline{\mathrm{g}}\overline{\partial}\underline{\mathrm{g}}^{-1}$, where $\underline{\mathrm{g}}$ denotes the operator corresponding to multiplication by g. Exactness at the center is simply the statement that

$$\underline{\mathrm{g}}\overline{\partial}\underline{\mathrm{g}}^{-1} = \underline{\mathrm{g}}'\overline{\partial}\underline{\mathrm{g}}'^{-1} \iff \overline{\partial}(g^{-1}g') = 0.$$

Surjectivity follows from a standard integrability theorem. For the proof, see [DK97] (2.1.53). An element $\overline{\partial} \in \mathrm{Hol}(n)$ transforms as

$$\overline{\partial} \mapsto \underline{u}\overline{\partial}\underline{u}^{-1} = \overline{\partial} - \underline{(\overline{\partial}u)u^{-1}},$$

which is the transformation law for a genuine $\overline{\partial}$ operator on the associated holomorphic vector bundle. Thus holomorphic structures correspond to $\overline{\partial}$ operators.

A.3.2 The gauge principle

Let $P \to X$ be a principal G-bundle, and let $\mathfrak{g}$ denote the Lie algebra of G. Recall that P has a right G-action. Let $m_R(\mathrm{g})$ denote the right action $p \mapsto p\mathrm{g}$. For any $\chi \in \mathfrak{g}$, let ℓ_χ denote the vector field on P generated by the flow $m_R(e^{-\chi t})$. The vector fields ℓ_χ generate the vertical tangent space T^vP of P, which is the kernel of $\pi_* : TP \to TM$. Observe that $\ell_\chi \mathrm{g}^{-1}$ is generated by the flow $p \mapsto p\mathrm{g}^{-1}e^{\mathrm{g}\chi\mathrm{g}^{-1}t}$, so $\ell_\chi \mathrm{g}^{-1} = \ell_{\mathrm{g}\chi\mathrm{g}^{-1}}$.

Inside of $\Omega^1(P; \pi^*(\mathrm{ad}_P))$ are the "basic" one-forms $\pi^*(\Omega^1(X; \mathrm{ad}_P))$, which pull back from X. Since $\pi_*(\ell_\chi) = 0$, we have $a(\ell_\chi) = 0$. A point in the adjoint bundle ad_P is an equivalence class of the form $[\![p, \chi]\!] \sim [\![p\mathrm{g}^{-1}, \mathrm{g}\chi\mathrm{g}^{-1}]\!]$. Thus

$$\pi^*(\mathrm{ad}_P) = \{(q, [\![p, \chi]\!]) \in P \times \mathrm{ad}_P \mid \pi(q) = \pi(p)\}.$$

Since a is a pullback, its ad_P values are constant in the fiber direction. Specifically, if $v_p \in T_pP$, then there is some $\chi \in \mathfrak{g}$ such that for all $\mathrm{g} \in G$,

$$a(v_p\mathrm{g}^{-1}) = (p\mathrm{g}^{-1}, [\![p, \chi]\!]) = (p\mathrm{g}^{-1}, [\![p\mathrm{g}^{-1}, \mathrm{g}\chi\mathrm{g}^{-1}]\!]).$$

We can identify $\pi^*(\mathrm{ad}_P)$ with T^vP via the isomorphism

$$(p, [\![p, \chi]\!]) \mapsto \ell_\chi \text{ at } p.$$

Under this identification, $a \in \Omega^1(P; T^vP)$ satisfies

$$a(v_p\mathrm{g}^{-1}) = \ell_{\mathrm{g}\chi\mathrm{g}^{-1}} = a(v_p)\mathrm{g}^{-1}.$$

In fact, we can identify $\Omega^1(X;\mathrm{ad}_P)$ with the set of $a \in \Omega^1(P;T^vP)$ satisfying

$$a(\ell_\chi) = 0 \text{ and } a(v\mathrm{g}) = a(v)\mathrm{g} \text{ for } v \in TP, \mathrm{g} \in G. \tag{A.35}$$

Definition A.3.5. The set of smooth connections $\mathcal{A}_P$ on P is

$$\mathcal{A}_P := \left\{A \in \Omega^1(P;T^vP) \mid A(\ell_\chi) = \ell_\chi, \text{ and } A(v\mathrm{g}) = A(v)\mathrm{g} \text{ for all } v \in TP, \mathrm{g} \in G\right\}.$$

The difference of any two connections $a = A' - A$ belongs to $\Omega^1(X;\mathrm{ad}_P)$ by the identification (A.35). In fact, $\mathcal{A}_P$ is an affine space modeled on $\Omega^1(X;\mathrm{ad}_P)$.

We will now define the associated connection $\rho(A)$ on any associated fiber bundle $E = P \times_\rho F$. Let $T^vE := \ker(\pi_*)$ denote the vertical tangent bundle of E. The connection $\rho(A)$ will be an element of $\Omega^1(E;T^vE)$. We will define it on $\Omega^1(P \times F;TF)$ and show that it passes to the quotient $P \times F \to E$. For a vector $v_p + v_f \in T_{(p,f)}P \times F$ we define

$$\rho(A)(v_p + v_f) := \rho_*(A(v_p)) + v_f.$$

Consider $(p,f) = (\llbracket x_\alpha, \mathrm{g}_\alpha \rrbracket, f)$. Then for $\rho(A)$ to descend to the quotient $P \times F \to P \times_\rho F$, it must vanish along the vector fields generated by

$$(\llbracket x_\alpha, \mathrm{g}_\alpha e^{-\chi t} \rrbracket, \rho(e^{\chi t})f).$$

We have $v_p = \ell_\chi$ and $v_f = \rho_*(-\ell_\chi)$, so indeed $\rho(A)(v_p + v_f) = 0$, and $\rho(A)$ descends to $\Omega^1(P \times F;T^vE)$.

If $E \to X$ is a fiber bundle $E = P \times_\rho F$ and $A \in \mathcal{A}_P$, then for $s \in \Omega^0(X;E)$, we define the *covariant derivative* $\nabla_A s := s^*(\rho(A)) \in \Omega^1(X;s^*(T^vE))$.

If ρ is left multiplication m_L on G, then $m_L(A) = A$. For a local section ϕ, we have $\nabla_A\phi \in \Omega^1(X;\phi^*(T^vP))$. If we view ϕ as an element of $\mathrm{Iso}_X(G,P)$, then $\phi^{-1}\nabla_A\phi \in \Omega^1(X;T_eG) = \Omega^1(X;\mathfrak{g})$. For any local trivialization ϕ, we define the *local connection one-form* $A_\phi := \phi^{-1}\nabla_A\phi$. For a local section $\phi \in \Omega^0(X;P)$ and $\mathrm{g} \in \Omega^0(X;G)$, we have the transformation rule

$$\begin{aligned}
A_{\phi\mathrm{g}} &= (\phi\mathrm{g})^{-1}\nabla_A(\phi\mathrm{g}) \\
&= (\phi\mathrm{g})^{-1}\left((\nabla_A\phi)\mathrm{g} + \phi d\mathrm{g}\right) \\
&= \mathrm{g}^{-1}(\phi^{-1}\nabla_A\phi)\mathrm{g} + \mathrm{g}^{-1}d\mathrm{g} \\
&= \mathrm{g}^{-1}A_\phi\mathrm{g} + \mathrm{g}^{-1}d\mathrm{g} \in \Omega^1(X;\mathfrak{g}).
\end{aligned}$$

Similarly,

$$A_{\phi\mathrm{g}^{-1}} = \mathrm{g}A_\phi\mathrm{g}^{-1} - (d\mathrm{g})\mathrm{g}^{-1} \in \Omega^1(X;\mathfrak{g}).$$

We now explain why this is called the connection one-form. If $s \in \Omega^1(X;E)$ is a section of a fiber bundle E, then $\phi^{-1}s \in \Omega^1(U;F)$ is the ϕ-trivialized section. We compute

$$\begin{aligned}
\nabla_A s &= \nabla_A(\phi\phi^{-1}s) \\
&= \phi d(\phi^{-1}s) + \rho(\nabla_A\phi)(\phi^{-1}s) \\
&= \phi\left(d + \rho(A_\phi)\right)(\phi^{-1}s).
\end{aligned}$$

Definition A.3.6. The group of *smooth bundle automorphisms*, or *gauge transformations* of a bundle P is

$$\mathcal{G}_P := \Omega^0(X; \mathrm{Ad}_P).$$

For any fiber bundle $E \to X$ associated to P, we get an action of $\mathcal{G}_P$ on sections $\Omega^0(X; E)$ given by

$$s \mapsto \rho(\mathrm{g})s.$$

For any operator $\underline{D} : \Omega^0(E_1) \to \Omega^0(E_2)$, the natural action is

$$\underline{D} \mapsto \rho_{E_2}(\mathrm{g})\underline{D}\rho_{E_1}(\mathrm{g}^{-1}).$$

For example,

$$\begin{aligned} \underline{\nabla_A} &\mapsto \mathrm{g}\underline{\nabla_A}\mathrm{g}^{-1} \\ &= \underline{\nabla_A} - (\nabla_A \mathrm{g})\mathrm{g}^{-1}, \end{aligned}$$

where $(\nabla_A \mathrm{g})\mathrm{g}^{-1} \in \Omega^1(X; \mathrm{ad}_P)$. This defines an action of $\mathcal{G}_P$ on $\mathcal{A}_P$ given by

$$A \mapsto \mathrm{g}(A) = A - (\nabla_A \mathrm{g})\mathrm{g}^{-1}.$$

If we examine this expression in a local frame ϕ and define $\mathrm{g}_\phi := \phi^{-1}\mathrm{g}$, then

$$\begin{aligned} A &= \phi(A_\phi - (d\mathrm{g}_\phi + A_\phi \mathrm{g}_\phi - \mathrm{g}_\phi A_\phi)\mathrm{g}_\phi^{-1}) \\ &= \phi(\mathrm{g}_\phi A_\phi \mathrm{g}_\phi^{-1} - d\mathrm{g}_\phi \mathrm{g}_\phi^{-1}) \\ &= \phi(A_{\phi \mathrm{g}_\phi^{-1}}). \end{aligned}$$

This illustrates the fact that locally, a gauge transformation is equivalent to a change of trivialization.

Definition A.3.7. The *gauge principle* is the observation that under the action $\mathcal{G}_P$ on both $\mathcal{A}_P$ and fiber bundles, expressions involving ∇_A and associated operators transform equivariantly.

A.3.3 Connections and differential operators

If we have a trivialization ϕ_α of P, then for the ϕ_α-trivialized forms $\Omega^\bullet(X; V_P)_\alpha$, we have

$$\Omega^\bullet(X; V_P)_\alpha \cong V \otimes \Omega^\bullet(X).$$

We extend our operators d, d^*, $\rho_W(R)$, the inner product $\cdot$, etc. from $\Omega^\bullet(X)$ to operators d_α, d_α^*, $\cdot$, $\rho_W(R)$, etc. which act only on $\Omega^\bullet(X)$, componentwise. (Note that differential operators depend on ϕ_α, while tensorial operators like $\cdot$ and $\rho_W(R)$ are independent of the choice of s_α, and thus don't require an α subscript.)

If we also have a frame $\{e_i\}$ for TX, then we similarly extend operators ι_i, ϵ^i, $\underline{\mathcal{L}}^0_{\alpha,i}$, $\underline{\mathcal{L}}_{\alpha,i}$, $\underline{\nabla}_{\alpha,i}$, etc. which act only on the $\Omega^\bullet(X)$ component of $V \otimes \Omega^\bullet(X)$. A change of frame G then acts on the

$\Omega^\bullet(X)$ component. Since all these operators act only on the form component, they are $\mathcal{G}$-invariant:

$$\mathrm{g}\left\{\iota_i\right\} = \iota_i, \qquad \mathrm{g}\left\{\epsilon^i\right\} = \epsilon^i, \qquad \mathrm{g}\left\{\underline{\mathcal{L}}^0_{\alpha,i}\right\} = \underline{\mathcal{L}}^0_{\alpha,i}, \qquad \mathrm{g}\left\{\underline{d}_\alpha\right\} = \underline{d}_\alpha, \qquad \mathrm{g}\left\{\underline{\nabla}_{\alpha,i}\right\} = \underline{\nabla}_{\alpha,i}, \qquad \dots$$

For an operator $\underline{D}$ to be $\mathcal{G}$-equivariant, it should satisfy

$$\mathrm{g}\left\{\underline{D}_\alpha\right\} = \mathrm{g}_\alpha\, \underline{D}_\alpha\, \mathrm{g}_\alpha^{-1}.$$

However, a differential operator such as $\underline{\partial}_{\alpha,i}$ satisfies

$$\begin{aligned}\mathrm{g}\left\{\underline{\partial}_{\alpha,i}\right\} &= \underline{\partial}_{\alpha,i} \\ &= \mathrm{g}_\alpha\, \underline{\partial}_{\alpha,i}\mathrm{g}_\alpha^{-1} + \left[\underline{\partial}_{\alpha,i}, \mathrm{g}_\alpha\right]\mathrm{g}_\alpha^{-1} \\ &= \mathrm{g}_\alpha\, \underline{\partial}_{\alpha,i}\mathrm{g}_\alpha^{-1} + \left(\partial_{\alpha,i}\mathrm{g}_\alpha\right)\mathrm{g}_\alpha^{-1},\end{aligned}$$

where $\partial_{\alpha,i}\mathrm{g}_\alpha$ at the point $x \in X$ is an element of $T_{\mathrm{g}_\alpha(x)}G$.

A connection A on P is equivalent to a collection of $\mathfrak{g}$-valued functions $\{A_{\alpha,i} \in \Omega^0(X;\mathfrak{g})\}$, which transform as

$$G(A)_{\alpha,i} = G^j{}_i A_{\alpha,j}, \text{ and } \mathrm{g}\left\{A_{\alpha,i}\right\} = \mathrm{g}_\alpha A_{\alpha,i}\mathrm{g}_\alpha^{-1} - (\partial_{\alpha,i}\mathrm{g}_\alpha)\mathrm{g}_\alpha^{-1}.$$

Given some connection A, we define the covariant derivative on $\Omega^\bullet(X;V_P)$ by

$$(\underline{\nabla}_{A,i})_\alpha := \underline{\nabla}_{\alpha,i} + A_{\alpha,i}.$$

We verify that $\underline{\nabla}_{A,i}$ is equivariant:

$$\begin{aligned}\mathrm{g}\left\{(\underline{\nabla}_{A,i})_\alpha\right\} &= \underline{\nabla}_{\alpha,i} + \mathrm{g}_\alpha A_{\alpha,i}\mathrm{g}_\alpha^{-1} - ((\partial_i\mathrm{g}_\alpha)\mathrm{g}_\alpha^{-1}) \\ &= \mathrm{g}_\alpha\, \underline{\nabla}_{\alpha,i}\, \mathrm{g}_\alpha^{-1} + \left[\underline{\nabla}_{\alpha,i}, \mathrm{g}_\alpha\right](\mathrm{g}_\alpha^{-1}) - ((\partial_i\mathrm{g}_\alpha)\mathrm{g}_\alpha^{-1}) \\ &= \mathrm{g}_\alpha\, \nabla_{\alpha,i}\, \mathrm{g}_\alpha^{-1}\,.\end{aligned}$$

Suppose V is a vector space equipped with a G-invariant inner product denoted by $\langle vw\rangle$ for $v, w \in V$. Given $B, C \in \Omega^\bullet(X, V_P)$, we define the inner product of B and C by

$$\int_X \langle B\cdot C\rangle := \int_X \langle B\cdot C\rangle\, d\mathrm{vol}_M.$$

Integration by parts for $\nabla_{A,i}$ is

$$\int_X \langle \nabla_{A,i}\alpha\cdot\beta\rangle = \int_{\partial M}\langle\alpha\cdot\beta\rangle\; a_i d\mathrm{vol}_M + \int_X \langle\alpha\cdot\nabla_{A,i}{}^*\beta\rangle\,,$$

where the formal metric adjoint is

$$\nabla_{A,i}{}^* = \nabla^*_{\alpha,i} + A^*_{\alpha,i}\,, \tag{A.36}$$

where $\nabla^*_{\alpha,i}$ is given by (A.31). Usually $A_{\alpha,i}$ will be antisymmetric, so $A^*_{\alpha,i} = -A_{\alpha,i}$.

A.3.4 Curvature of principal bundles

Without loss of generality we will fix some trivialization ϕ_α. Since the subscript α is fixed, it becomes somewhat redundant, though it still serves the purpose of distinguishing between a section itself and its components. We omit the subscript α with the hope that the distinction will be clear from context.

Define the curvature components

$$(F_A)_{ij} := \mathcal{L}_i A_j - \mathcal{L}_j A_i + \left[A_i, A_j\right] - c_i{}^k{}_j A_k.$$

This expression arises naturally from

$$\begin{aligned} &\nabla_{A,i}\nabla_{A,j} - \nabla_{A,j}\nabla_{A,i} - c_i{}^k{}_j\nabla_{A,k} \\ &= (\nabla_i + A_i)(\nabla_j + A_j) - (\nabla_j + A_j)(\nabla_i + A_i) - c_i{}^k{}_j(\nabla_k + A_k) \\ &= R_{ijk\ell}\epsilon^k\iota_\ell + \left[\nabla_i, A_j\right] - \left[\nabla_j, A_i\right] + \left[A_i, A_j\right] - c_i{}^k{}_j A_k \\ &= R_{ijk\ell}\epsilon^k\iota_\ell + (F_A)_{ij}. \end{aligned} \tag{A.37}$$

The components $(F_A)_{ij}$ transform as $\mathrm{g}\left\{(F_A)_{ij}\right\} = \mathrm{g}(F_A)_{ij}\mathrm{g}^{-1}$, so $(F_A)_{ij} \in \Omega^0(M;\mathrm{ad}_P)$, where ad is the G-vector space $\mathfrak{g}$ equipped with the adjoint action. Furthermore, under a change of frame, $G((F_A)_{ij}) = G^{i'}{}_i G^{j'}{}_j (F_A)_{ij}$, so $F_A = \frac{1}{2}(F_A)_{ij}e^{ij} \in \Omega^2(M;\mathrm{ad}_P)$.

We define the equivariant operator

$$d_A := \epsilon^i\nabla_{A,i}.$$

If A is a metric connection, so that $A_i^* = -A_i$, then its formal metric adjoint is

$$d_A^* = (\nabla_i^* + A_i^*)\iota^i = -\iota_i\nabla_i - \iota^i A_i = -\iota_i\nabla_{A,i}.$$

We verify the standard result

$$\begin{aligned} (d_A)^2 &= \epsilon^i\nabla_{A,i}\epsilon^j\nabla_{A,j} \\ &= \epsilon^{ij}\nabla_{A,i}\nabla_{A,j} + \epsilon^i\left[\nabla_{A,i}, \epsilon^j\right]\nabla_{A,j} \\ &= \frac{1}{2}\epsilon^{ij}\left(\nabla_{A,i}\nabla_{A,j} - \nabla_{A,j}\nabla_{A,i} - c_i{}^k{}_j\nabla_{A,k}\right) \\ &= \frac{1}{2}\left(R_{ijk\ell}\epsilon^{ijk}\iota_\ell + \epsilon^{ij}(F_A)_{ij}\right) \\ &= F_A \wedge . \end{aligned}$$

The Riemannian curvature cancels because of the first Bianchi identity (A.29).

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Appendix B

Fundamental computations

B.1 Weitzenböck formulas for real differential forms

Theorem B.1.1 (Weitzenböck formula for forms)**.** *As operators on differential forms,*

$$\nabla^*\nabla = d^*d + dd^* + \rho_{\mathrm{W}}(R), \tag{B.1}$$

where

$$\rho_{\mathrm{W}}(R) := R_{ijk\ell}\,\epsilon^i\iota^j\epsilon^k\iota^\ell, \tag{B.2}$$

and the operators ϵ and ι are defined in A.1.4.

In the proof, we will use the standard Clifford algebra representation (A.10) on differential forms. The associated Dirac operator $\not\partial$ is

$$\not\partial = \gamma^i\nabla_i = \epsilon^i\nabla_i - \iota^i\nabla_i = d + d^*,$$

and

$$\not\partial^2 = d^*d + dd^*. \tag{B.3}$$

Lemma B.1.2. *The curvature term $\rho_{\mathrm{W}}(R)$ is*

$$\rho_{\mathrm{W}}(R) = -\tfrac{1}{2}\gamma^{ij}\left(\left[\nabla_i, \nabla_j\right] - c_i{}^k{}_j\nabla_k\right). \tag{B.4}$$

Proof. First note that $\rho_{\mathrm{W}}(R) = -\frac{1}{2}\gamma^{ij}R_{ij}$, since

$$\begin{aligned}
-\frac{1}{2}\gamma^{ij}R_{ij} &= -\frac{1}{2}(\epsilon^i - \iota^i)(\epsilon^j - \iota^j)R_{ijk\ell}\epsilon^k\iota^\ell \\
&= \frac{1}{2}(\epsilon^i\iota^j + \iota^i\epsilon^j)R_{ijk\ell}\epsilon^k\iota^\ell - \frac{1}{2}\cancel{\epsilon^{ijk}R_{ijk\ell}}\iota^\ell + \frac{1}{2}\cancel{\iota^{ij\ell}R_{ijk\ell}}\epsilon^k \\
&= \rho_{\mathrm{W}}(R),
\end{aligned}$$

the extraneous curvature terms cancelling by the first Bianchi identity (A.29). Finally, note that

$$R_{ij} = \left[\nabla_i, \nabla_j\right] - c_i{}^k{}_j \nabla_k.$$

□

Proof of Theorem B.1.1. We compute

$$\begin{aligned}
& d^* d + d d^* - \nabla^* \nabla + \rho_{\mathrm{W}}(R) \\
&= \not{\partial}^2 - \nabla^* \nabla - \frac{1}{2} \gamma^{ij} \left(\left[\nabla_i, \nabla_j \right] - c_i{}^k{}_j \nabla_k \right) \qquad \text{(B.5)} \\
&= \gamma^i \nabla_i \gamma^j \nabla_j + (\nabla_i + \Gamma_k{}^k{}_i) g^{ij} \nabla_j - \frac{1}{2} \gamma^{ij} \left(\nabla_{ij} - \nabla_{ji} - c_i{}^k{}_j \nabla_k \right) \\
&= (\cancel{\gamma^{ij} \nabla_{ij}} - \gamma^{ik} \Gamma_i{}^j{}_k \nabla_j) + \left((\partial_i g^{ij}) \nabla_j + \cancel{g^{ij} \nabla_{ij}} + \Gamma_k{}^{kj} \nabla_j \right) \\
& \qquad \cancel{- \frac{1}{2} (\gamma^{ij} - \gamma^{ji})} \nabla_{ij} + \frac{1}{2} \gamma^{ij} (\Gamma_i{}^k{}_j - \Gamma_j{}^k{}_i) \nabla_k \\
&= (-\gamma^{ik} \Gamma_i{}^j{}_k - (\cancel{\Gamma_i{}^{ij}} + \Gamma_i{}^{ji}) + \cancel{\Gamma_k{}^{kj}} + \frac{1}{2} (\gamma^{ik} - \gamma^{ki}) \Gamma_i{}^j{}_k) \nabla_j \\
&= (-\gamma^{ik} + \frac{1}{2} (\gamma^{ik} - \gamma^{ki})) \Gamma_i{}^j{}_k - \Gamma_i{}^{ji} \\
&= 0.
\end{aligned}$$

□

Let X be an oriented Riemannian manifold with boundary $Y = \partial X$, and let $i : Y \hookrightarrow X$ be the inclusion map. Let $\Omega^\bullet(X)|_Y$ denote the pullback $i^* \Omega^\bullet(X)$. We identify $\Omega^\bullet(Y)$ with a subspace of $\Omega^\bullet(X)|_Y$; specifically if e_η is the unit outward normal along Y, then $\Omega^\bullet(Y) \cong \ker a_\eta$. The natural complement is im a_η, which we identify with $\Omega^{\bullet-1}(Y)$. This gives us the decomposition

$$\Omega^\bullet(X)|_Y \cong \Omega^\bullet(Y) \oplus \Omega^{\bullet-1}(Y). \qquad \text{(B.6)}$$

Explicitly, if $\beta \in \Omega^\bullet(X)$, then

$$\beta|_Y = \beta^\parallel + e^\eta \wedge \beta^\perp, \qquad \text{(B.7)}$$

where

$$\beta^\parallel = i^*(\beta), \text{ and } \beta^\perp = i^*(a_\eta \beta) = (-1)^{n(p+1)} \star i^*(\star \beta).$$

Let $\{e_i\}_{i=1}^{n-1}$ be a frame for Y. Let $\nabla^\parallel$ denote the Levi-Civita connection on $\Omega^\bullet(Y)$. We may extend it to $\Omega^\bullet(X)|_Y$ by the rule $\nabla^\parallel e^\eta := 0$. It is related to the Levi-Civita connection ∇ on M by

$$\nabla_i = \nabla_i^\parallel + N_{ij} \left(\epsilon^\eta \iota_j - \epsilon^j \iota_\eta \right), \qquad \text{(B.8)}$$

where $N_{ij} := e^\eta \cdot \nabla_i e_j$ is the second fundamental form. (For the boundary of a Euclidean ball, the second fundamental form is negative-definite due to the outward choice of conormal.) The mean curvature of Y is defined by

$$H := \sum_{i=1}^{n-1} N_i^i. \tag{B.9}$$

An important application of (B.8) are the formulas

$$\begin{aligned} (d\beta)|_Y &= d\beta^{\parallel} + dt \wedge \left(\dot{\beta}^{\parallel} - N\beta^{\parallel} - d\beta^{\perp}\right), \\ (d^*\beta)|_Y &= (-\dot{\beta}^{\perp} + d^*\beta^{\parallel} + N\beta^{\perp}) + dt \wedge -d^*\beta^{\perp}. \end{aligned} \tag{B.10}$$

Theorem B.1.3 (Integrated Weitzenböck formula for forms)**.** *If X is a manifold with boundary $\partial X = Y$, for any differential forms $\alpha, \beta \in \Omega^\bullet(X)$,*

$$\begin{aligned} \int_X \nabla\alpha \cdot \nabla\beta = \int_X (d\alpha \cdot d\beta + d^*\alpha \cdot d^*\beta + \alpha \cdot \rho_{\mathrm{W}}(R)\beta) + \\ + \int_Y \left(\alpha^{\parallel} \cdot d^{\parallel}\beta^{\perp} + \alpha^{\perp} \cdot d^{*\parallel}\beta^{\parallel} + \alpha^{\parallel} \cdot N\beta^{\parallel} + \alpha^{\perp} \cdot (H - N)\beta^{\perp}\right), \end{aligned} \tag{B.11}$$

where ρ_W is (B.2)*, $\alpha^{\parallel}$ and $\alpha^{\perp}$ are respectively the parallel and perpendicular components of α along the boundary* (B.7)*, N is the second fundamental form acting as a derivation (Definition A.1.3), and H is the mean curvature* (B.9).

Proof. Again we will use the standard Clifford algebra representation (A.10) on differential forms. From (B.1), we know that

$$\int_X \left\langle \alpha \cdot \left(\partial\!\!\!/^2 - \nabla^*\nabla + \rho_{\mathrm{W}}(R)\right)\beta \right\rangle = 0.$$

Adding this to the left hand side of (B.11), we get a sum of two boundary terms

$$\int_X \left\langle (\nabla\alpha \cdot \nabla\beta - \alpha \cdot \nabla^*\nabla\beta) + \left(\alpha \cdot \partial\!\!\!/^2\beta - \partial\!\!\!/\alpha \cdot \partial\!\!\!/\beta\right)\right\rangle.$$

It remains to show that these combine to give the right hand side of (B.11). The first boundary term was computed in (A.32). For the second boundary term, we use (A.30) to compute

$$\begin{aligned} \int_Y \gamma^i\alpha \cdot \partial\!\!\!/\beta \, \iota_i d\mathrm{vol}_Y &= \int_X \left\langle \nabla_i\gamma^i\alpha \cdot \partial\!\!\!/ + \gamma^i\alpha \cdot \nabla_i\partial\!\!\!/\beta + \Gamma_j{}^j{}_i\gamma^i\alpha \cdot \partial\!\!\!/\beta\right\rangle \\ &= \int_X \left\langle \partial\!\!\!/\alpha \cdot \partial\!\!\!/\beta - \alpha \cdot \partial\!\!\!/^2\beta\right\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_X \left\langle (\nabla\alpha \cdot \nabla\beta - \alpha \cdot \nabla^*\nabla\beta) + \left(\alpha \cdot \partial\!\!\!/^2\beta - \partial\!\!\!/\alpha \cdot \partial\!\!\!/\beta\right)\right\rangle \\ = &\int_Y \left(\alpha \cdot g^{ij}\nabla_j\beta - \gamma^i\alpha \cdot \partial\!\!\!/\beta\right) \iota_i d\mathrm{vol}_X \\ = &\int_Y \left(\alpha \cdot (g^{ij} + \gamma^{ij})\nabla_j\beta\right) \iota_i d\mathrm{vol}_X. \end{aligned}$$

Consider an orthonormal coframe $\{e^\eta, e^1, e^2, \ldots, e^{n-1}\}$ along ∂M, where e^η is the outward unit conormal. The volume form $d\mathrm{vol}_Y$ on Y is $\iota_\eta d\mathrm{vol}_X$. The above integral reduces to

$$\int_Y \left\langle \alpha \cdot \left(g^{\eta i} + \gamma^{\eta i}\right) \nabla_i \beta \right\rangle .$$

Since the $i = \eta$ term vanishes, we assume $i \neq \eta$, which allows us to cancel the $g^{\eta i}$ term and to get

$$\begin{aligned}
&= \int_X \left\langle -\gamma^\eta \alpha \cdot \gamma^i \nabla_i \beta \right\rangle \\
&= \int_X \left\langle -\gamma^\eta \alpha \cdot \gamma^i (\nabla_i^\| + N_{ij} \epsilon^\eta \iota_j) \beta \right\rangle \\
&= \int_X \left\langle -\gamma^\eta \alpha \cdot (\partial\!\!\!/^{\|} + \gamma^i (N_{ij} \epsilon^\eta \iota_j - N_{ij} \epsilon^j \iota_\eta)) \beta \right\rangle .
\end{aligned}$$

Now we express α and β in terms of parallel and perpendicular components:

$$\begin{aligned}
\gamma^\eta \alpha &= -\alpha^\perp + e^\eta \wedge \alpha^\|, \\
\partial\!\!\!/^{\|} \beta &= (d^\| + d^{*\|})(\beta^\| + e^\eta \wedge \beta^\perp), \\
\gamma^\eta \alpha \cdot \partial\!\!\!/^{\|} \beta &= -\alpha^\perp \cdot d^{\|*} \beta^\| - \alpha^\| \cdot d^\| \beta^\perp, \\
\alpha \cdot \gamma^\eta \gamma^i (N_{ij} \epsilon^\eta \iota_j - N_{ij} \epsilon^j \iota_\eta)) \beta &= \alpha \cdot ((-\iota^\eta) \epsilon^i N_{ij} \epsilon^\eta \iota_j \beta^\| - \epsilon^\eta (-\iota^i) N_{ij} \epsilon^j \beta^\perp) \\
&= N_{ij} \alpha \cdot (\epsilon^i \iota_j \beta^\| + e^\eta \wedge \iota^i \epsilon^j \beta^\perp) \\
&= \alpha^\| \cdot N \beta^\| + \alpha^\perp \cdot (H - N) \beta^\perp.
\end{aligned}$$

Putting everything together, we get the desired boundary term for (B.11). □

B.2 Weitzenböck formula for bundle-valued differential forms

Theorem B.2.1 (Weitzenböck formulae for bundle-valued forms)**.** *Let M be an oriented manifold with boundary. For any associated vector bundle $E \to M$ with a metric connection A,*

$$\nabla_A^* \nabla_A = d_A^* d_A + d_A d_A^* + \rho_{\mathrm{W}}(R) + F_A, \tag{B.12}$$

where ρ_W is (B.2)*, F_A acts on form components as a derivation (Definition A.1.3) and with the associated action on E-components. Furthermore, for any $B_1, B_2 \in \Omega^\bullet(M; E)$,*

$$\begin{aligned}
\int_X \langle \nabla_A B_1 \cdot \nabla_A B_2 \rangle = \int_X \langle B_1 \cdot (\rho_{\mathrm{W}}(R) + F_A) B_2 + d_A B_1 \cdot d_A B_2 + d_A^* B_1 \cdot d_A^* B_2 \rangle + \\
+ \int_Y \left\langle B_1^\| \cdot d_A^\| B_2^\perp + B_1^\perp \cdot d_A^{*\|} B_2^\| + B_1^\| \cdot N B_2^\| + B_1^\perp \cdot (H - N) B_2^\perp \right\rangle,
\end{aligned} \tag{B.13}$$

where $\alpha^\|$ and $\alpha^\perp$ are respectively the parallel and perpendicular components of α along the boundary (B.7)*, N is the second fundamental form acting as a derivation (Definition A.1.3), and H is the mean curvature* (B.9).

Upon substituting $\nabla \mapsto \nabla_A$, the proofs of Theorem B.1.1 and Theorem B.1.3 work with minor modifications. First, note that we must replace (B.3) with

$$\partial\!\!\!/_A^2 = d_A^* d_A + d_A d_A^* + (F_A \wedge) + (F_A \vee).$$

Also, we must modify (B.4) in accordance with (A.37). The extra term is

$$\begin{aligned}-\tfrac{1}{2}\gamma^{ij}(F_A)_{ij} &= \epsilon^i \iota^j (F_{A^*})_{ij} - \tfrac{1}{2}(F_{A^*})_{ij}\epsilon^{ij} - \tfrac{1}{2}(F_{A^*})_{ij}\iota^{ij} \\ &= F_A - (F_A\wedge) - (F_A\vee).\end{aligned}$$

The analogue of (B.4) is thus

$$-\tfrac{1}{2}\gamma^{ij}\left(\left[\nabla_{A,i}, \nabla_{A,j}\right] - c_i{}^k{}_j \nabla_{A,k}\right) = \rho_{\mathrm{W}}(R) + F_A - (F_A\wedge) - (F_A\vee).$$

Consequently,

$$d_A^* d_A + d_A d_A^* - \nabla_A^* \nabla_A + \rho_{\mathrm{W}}(R) + F_A = \partial\!\!\!/_A^2 - \nabla_A^*\nabla_A - \frac{1}{2}\gamma^{ij}\left(\left[\nabla_{A,i}, \nabla_{A,j}\right] - c_i{}^k{}_j \nabla_{A,k}\right),$$

and the same computation (B.5) proves (B.12) after adding the subscript A to all differential operators.

Similarly, to get (B.13), the same proof of Theorem B.1.3 applies after adding subscripts of A.

B.3 Representations of Riemannian curvature

The goal of this section is to prove Theorem B.3.2 about how the Weitzenbock representation of the Riemannian curvature acts on differential forms.

An *algebraic curvature tensor* is a rank four tensor satisfying (A.29). Associated to any algebraic curvature tensor R are the tensors

$$\mathrm{Ric}_{ik} := R_{ijk\ell}g^{j\ell}, \qquad s := \mathrm{Ric}_{ij}g^{ij}, \qquad \mathrm{Ric}^0_{ik} := \mathrm{Ric}_{ik} - \frac{s}{n}g_{ik}.$$

The Kuklari-Nomizu product $\overline{\wedge}$ produces an algebraic curvature tensor $u \mathbin{\overline{\wedge}} v$ from two symmetric rank two tensors u and v by the formula

$$(u \mathbin{\overline{\wedge}} v)_{ijk\ell} := u_{ik}v_{j\ell} + u_{j\ell}v_{ik} - u_{i\ell}v_{jk} - u_{jk}v_{i\ell}.$$

We may decompose R as

$$R = W + \frac{1}{n-2}\left(\mathrm{Ric} - \frac{s}{2(n-1)}g\right) \mathbin{\overline{\wedge}} g, \tag{B.14}$$

where W is the Weyl tensor, which is defined by the above relation, and satisfies

$$W_{ijk\ell}g^{j\ell} = 0 \text{ for all } i, k.$$

We define an action of R on forms by

$$\rho_{\mathrm{N}}(R) := -\frac{1}{4}R_{ijk\ell}\epsilon^{ij}\iota_{k\ell}, \tag{B.15}$$

and is characterized by the property

$$e^{ij} \cdot \rho_{\mathrm{N}}(R) e^{kl} = R_{ijkl}.$$

The subscript N stands for "normal ordered," in reference to the factor $\epsilon^{ij}\iota_{k\ell}$ from the fermionic oscillator algebra.

Lemma B.3.1. *The representation $\rho_{\mathrm{W}}(R)$ defined in* (B.2) *is related to $\rho_{\mathrm{N}}(R)$ by*

$$\rho_{\mathrm{W}}(R) = 2\rho_{\mathrm{N}}(R) - \mathrm{Ric},$$

where Ric *acts as a derivation, as in Definition A.1.3.*

Proof. Working in an orthonormal frame so that $g_{ij} = \delta_{ij}$,

$$\begin{aligned}
2\rho_{\mathrm{N}}(R) &= -\frac{1}{2} R_{ijk\ell}\epsilon^{ij}\iota^{k\ell} \\
&= \frac{1}{2}\left(R_{ik\ell j} + R_{i\ell jk}\right)\epsilon^{ij}\iota^{k\ell} \\
&= \frac{1}{2} R_{ijk\ell}(\epsilon^{i\ell}\iota^{jk} + \epsilon^{ik}\iota^{\ell j}) \\
&= -R_{ijk\ell}\epsilon^{ik}\iota^{j\ell} \\
&= R_{ijk\ell}\epsilon^{i}\iota^{j}\epsilon^{k}\iota^{\ell} - R_{ijj\ell}\epsilon^{i}\iota^{\ell} \\
&= \rho_{\mathrm{W}}(R) + \rho_{\mathrm{D}}(\mathrm{Ric}).
\end{aligned}$$

□

Theorem B.3.2. *Let M be a Riemannian manifold of dimension n. Then the action $\rho_{\mathrm{W}}(R)$ on $\Omega^p(M)$ decomposes as*

$$\begin{aligned}
\rho_{\mathrm{W}}(R) &= 2\rho_{\mathrm{N}}(W) - \frac{n-2p}{n-2} Ric - \frac{p(p-1)}{(n-1)(n-2)} s \\
&= 2\rho_{\mathrm{N}}(W) - \frac{n-2p}{n-2} Ric^0 - \frac{p(n-p)}{n(n-1)} s.
\end{aligned} \tag{B.16}$$

Proof. The second equality follows from the first via the identity

$$\mathrm{Ric} = \mathrm{Ric}^0 + \frac{p}{n} s.$$

To obtain the first equality, we combine Lemma B.3.1 with (B.14), and use the identities

$$\rho_{\mathrm{N}}(\mathrm{Ric} \owedge g) = (p-1)\mathrm{Ric}, \qquad \rho_{\mathrm{N}}(g \owedge g) = p(p-1),$$

to compute

$$
\begin{aligned}
\rho_{\mathrm{W}}(R) &= 2\rho_{\mathrm{N}}(R) - \mathrm{Ric} \\
&= 2\rho_{\mathrm{N}}(W) + \frac{2}{n-2}\left(\rho_{\mathrm{N}}(\mathrm{Ric} \mathbin{\overline{\wedge}} g) - \frac{s}{2(n-1)}\rho_{\mathrm{N}}(g \mathbin{\overline{\wedge}} g)\right) - \mathrm{Ric} \\
&= 2\rho_{\mathrm{N}}(W) - \frac{n-2p}{n-2}\mathrm{Ric} - \frac{p(p-1)}{(n-1)(n-2)}s.
\end{aligned}
$$

□

B.4 Weyl curvature on a four-manifold

On a four-manifold, the Weyl curvature decomposes into two components $W = W^+ \oplus W^-$ which act via

$$\rho_{\mathrm{N}}(W^\pm) \in \Omega^0(X; \mathrm{End}_{\mathrm{Sym}^2_0}(\Lambda^{2,\pm}T^*X))$$

as symmetric traceless endomorphisms [AHS78], where ρ_{N} is defined by (B.15). This representation is faithful, so we make it implicit and declare

$$W^\pm \in \Omega^0(X; \mathrm{End}_{\mathrm{Sym}^2_0}(\Lambda^{2,\pm}T^*X)).$$

This defines a quadratic form on $B \in \Lambda^{2,\pm}T^*X$ given by

$$B \cdot W^\pm B.$$

We define an operator $\odot$ such that

$$B \cdot W^\pm B = W^\pm \cdot (B \odot B).$$

More generally, for any Euclidean vector space V, we define the bilinear operation

$$
\begin{aligned}
\odot : V \times V &\to \mathrm{End}_{\mathrm{Sym}^2_0}(V), \\
v \odot w &:= \pi_{\mathrm{Sym}^2_0}(v \otimes w^*).
\end{aligned}
$$

For example, $\rho_{\mathrm{W}}(R)$ (B.16) acting on $\Omega^{2,+}$ is

$$2W^+ - \tfrac{1}{3}s,$$

and induces the bilinear form

$$B \cdot \rho_{\mathrm{W}}(R)B = -\tfrac{1}{3}s\,|B|^2 + 2W^+ \cdot (B \odot B).$$

B.5 The Weitzenböck formula for self-dual two-forms

In this section, we specialize (B.13) to the case $n = 4$ with an adjoint-valued self-dual two-form $B = B_1 = B_2 \in \Omega^{2,+}(M; \mathrm{ad}_P)$.

Note that

$$F_A B = \left[(F_A)_{ij}, \epsilon^i \iota^j B\right] = -\iota_k \left[\tfrac{1}{2}(F_A)_{ij}, \epsilon^{ij}\iota_k B\right] = \left[F_A \centerdot B\right] = \left[F_A^+ \centerdot B\right],$$

and

$$B^\perp = \star B^\parallel, \qquad d_A B = - \star d_A^* B, \qquad \langle B \cdot \rho_W(R) B\rangle = -\tfrac{1}{3} s \left|B\right|^2 + 2W^+ \cdot \langle B \odot B\rangle,$$

where $\odot$ is the traceless symmetric product defined in Section B.4. Thus our equation (B.13) becomes

$$\left\|\nabla_A B\right\|^2 = \int_X \langle B \cdot \rho_W(R) B + B \cdot [F_A^+ \centerdot B]\rangle + \left\|d_A B\right\|^2 + \left\|d_A^* B\right\|^2 + \\ + 2\int_Y \left\langle B^\parallel \cdot d_{A^\parallel} \star B^\parallel + B^\parallel \cdot N B^\parallel \right\rangle.$$

Dividing by four, performing some substitutions, using (A.25) to get $\langle B \cdot [F_A^+ \centerdot B]\rangle = \langle F_A^+ \cdot [B \centerdot B]\rangle$, and making ρ_N implicit, we obtain

$$\tfrac{1}{2}\left\|d_A^* B\right\|^2 = \tfrac{1}{4}\left\|\nabla_A B\right\|^2 + \tfrac{1}{12}\int_X \left(s\left|B\right|^2 - 6W^+ \cdot \langle B \odot B\rangle\right) + \\ - \tfrac{1}{4}\int_X \langle F_A^+ \cdot [B \centerdot B]\rangle - \tfrac{1}{2}\int_Y \left\langle B^\parallel \cdot \left(d_{A^\parallel} \star + N\right) B^\parallel \right\rangle. \tag{B.17}$$

The unintegrated Weitzenbock formula will also be useful. Note that (B.12) becomes

$$\nabla_A^* \nabla_A B = 2 d_A d_A^* B + \left(-\tfrac{1}{3}s + 2W^+\right) B + \left[F_A \centerdot B\right]. \tag{B.18}$$

In particular, note that we can obtain an expression for the Laplacian of $\left|B\right|^2$ as

$$\begin{aligned} \tfrac{1}{8}\Delta\left|B\right|^2 + \tfrac{1}{4}\left|\nabla_A B\right|^2 &= \tfrac{1}{4}\langle B \cdot \nabla_A^* \nabla_A B\rangle \\ &= \tfrac{1}{2}\langle B \cdot d_A d_A^* B\rangle - \tfrac{1}{12}\left(s\left|B\right|^2 - 6W^+ \cdot \langle B \odot B\rangle\right) + \tfrac{1}{4}\langle B \cdot [F_A^+ \centerdot B]\rangle. \end{aligned} \tag{B.19}$$

Appendix C

Chern-Simons theory

C.1 An overview of Chern-Simons theory and topology

On a Lie group G, for any $\chi \in \mathfrak{g}$ let ℓ_χ denote the left-invariant vector field which generates the flow

$$\ell_\chi := \frac{d}{dt} R_{e^{t\chi}} \bigg|_{t=0},$$

where R_g denotes right-multiplication by $g \in G$. Define the tautological $\mathfrak{g}$-valued one-form $\alpha \in \Omega^1(G;\mathfrak{g})$ by

$$\alpha \cdot \ell_\chi = \chi.$$

By the conventions for the metric $\langle\rangle$ on $\mathfrak{g}$ from (A.18), if α is the tautological $\mathfrak{g}$-valued one-form on G, then

$$\langle \alpha \wedge [\alpha \wedge \alpha] \rangle \in \Omega^3(G)$$

represents $24\pi^2$ times an element of $H^3(G;\mathbb{Z})$ when $G = \mathrm{Sp}(1)$. Closedness follows from $d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$. To compute the normalization, we choose a basis $\{\chi_k\}$ for $\mathfrak{g}$ with dual basis $\{e^i\}$. We compute

$$\langle \alpha \wedge [\alpha \wedge \alpha] \rangle = \langle \chi_k \chi_\ell \rangle \, c_i{}^\ell{}_j e^{ijk}.$$

In the case of $\mathrm{Sp}(1)$, we use $\chi_1 = \mathbf{i}$, $\chi_2 = \mathbf{j}$, $\chi_3 = \mathbf{k}$ so that $c_i{}^\ell{}_j = 2\varepsilon_{ij\ell}$ and $\langle \chi_i \chi_j \rangle = 1$. Now $\varepsilon_{ijk} e^{ijk} = 6e^{123}$, where the e^i are dual to the χ_i. The left-invariant vector fields corresponding to χ_i are orthonormal on the unit quaternions, so $\int_{\mathrm{Sp}(1)} e^{123} = \mathrm{vol}(\mathbb{S}^3) = 2\pi^2$. Putting everything together, we compute

$$\int_{\mathrm{Sp}(1)} \langle \alpha \wedge [\alpha \wedge \alpha] \rangle = 24\pi^2.$$

In the Leray-Serre spectral sequence for the fibration $\mathrm{Sp}(1) \to E\mathrm{Sp}(1) \to B\mathrm{Sp}(1)$ with integral coefficients, we see that $(24\pi^2)^{-1} \langle \alpha \wedge [\alpha \wedge \alpha] \rangle$ represents a generator of $H^3(\mathrm{Sp}(1);\mathbb{Z})$ which transgresses to the second Chern class $c_2 \in H^4(B\mathrm{Sp}(1);\mathbb{Z})$.

The constant for $\mathrm{SO}(3)$ differs slightly from $\mathrm{Sp}(1)$. Again we choose χ_i so that $c_i{}^\ell{}_j = 2\varepsilon_{ij\ell}$ and

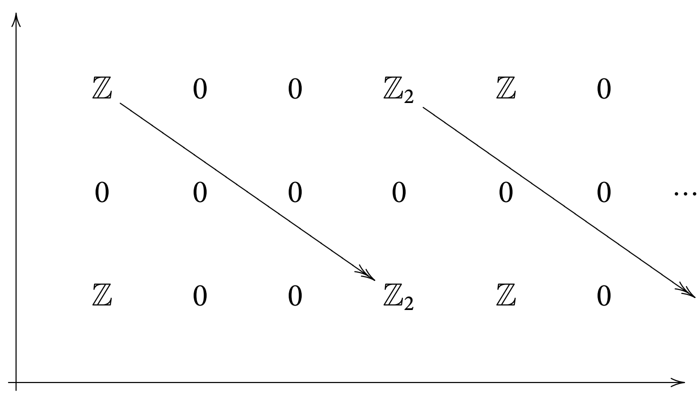
$\langle \chi_i \chi_j \rangle = 1$. Note that $\int_{\mathrm{SO}(3)} e^{123} = \mathrm{vol}(\mathbb{RP}^3) = \pi^2$, so

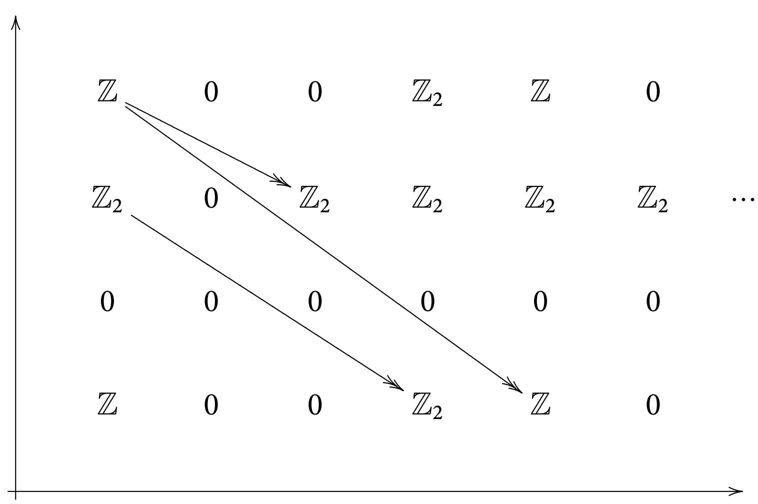
$$\int_{\mathrm{SO}(3)} \langle \alpha \wedge [\alpha \wedge \alpha] \rangle = 12\pi^2.$$

The class $(24\pi^2)^{-1} \langle \alpha \wedge [\alpha \wedge \alpha] \rangle$ transgresses to $-\frac{1}{4} p_1 \in H^4(B\mathrm{SO}(3); \mathbb{R})$. To understand this $\frac{1}{4}$ factor, we examine the fibration $\mathrm{SO}(3) \to E\mathrm{SO}(3) \to B\mathrm{SO}(3)$, and its $\mathrm{SO}(2)$-quotient $S^2 \to B\mathrm{SO}(2) \to B\mathrm{SO}(3)$. Though we will need them only through degree five, the cohomology rings of $B\mathrm{SO}(3)$ are

$$\begin{aligned} H^\bullet(B\mathrm{SO}(3); \mathbb{Z}) &= \mathbb{Z}[e, p_1]/2e, \\ H^\bullet(B\mathrm{SO}(3); \mathbb{Z}_2) &= \mathbb{Z}_2[w_2, w_3], \end{aligned}$$

where $e \in H^3(B\mathrm{SO}(3); \mathbb{Z})$ is the universal Euler class, $p_1 \in H^4(B\mathrm{SO}(3); \mathbb{Z})$ is the universal Pontryagin class, and $w_i \in H^i(B\mathrm{SO}(3); \mathbb{Z}_2)$ are the Stiefel-Whitney classes. Even without this foreknowledge, together the two Leray-Serre spectral sequences of these fibrations easily determine $H^\bullet(B\mathrm{SO}(3); \mathbb{Z})$ through degree five:





We observe that under d_2, the elements of $H^3(\mathrm{SO}(3); \mathbb{Z})$ not divisible by two must hit

$$w_2 \in H^2(B\mathrm{SO}(3); H^2(\mathrm{SO}(3))) \cong \mathbb{Z}_2.$$

The surviving even elements of $H^3(\mathrm{SO}(3);\mathbb{Z})$ under d_4 must hit the integral multiples of p_1. If $2\mathbb{Z}$ denotes the kernel of d_2, then the isomorphism $d_4 : 2\mathbb{Z} \to \mathbb{Z}$ naturally carries a factor of $\pm\frac{1}{2}$. This explains why half an integral class can transgress to a quarter of an integral class.

Let P be a principal bundle, with connection $A \in \mathcal{A}(P) \subset \Omega^1(P; \mathfrak{g}_P)$. The restriction of A to any fiber is α. Using the Leray-Serre filtration for de Rham cohomology, we transgress $(24\pi^2)^{-1} \langle \alpha \wedge [\alpha \wedge \alpha] \rangle$ by

$$
\begin{tikzcd}
\langle A \wedge [A \wedge A] \rangle \arrow[drr] & 0 & 0 & 0 & 0 \\
0 & 0 & \begin{array}{c} 3 \langle F_A \wedge [A \wedge A] \rangle \\ -3 \langle F_A \wedge [A \wedge A] \rangle \end{array} & 0 & 0 \\
0 & 0 & -6 \langle A \wedge F_A \rangle \arrow[u] \arrow[drr] & 0 & 0 \\
0 & 0 & 0 & 0 & -6 \langle F_A \wedge F_A \rangle
\end{tikzcd}
$$

This motivates us to define the Chern-Simons form

$$\mathrm{CS}(A) := \langle A \wedge (\tfrac{1}{6} [A \wedge A] - F_A) \rangle \in \Omega^3(P).$$

On fibers, $\mathrm{CS}(A)$ restricts to $\frac{1}{6} \langle \alpha \wedge [\alpha \wedge \alpha] \rangle$, and $d\mathrm{CS}(A) = -\langle F_A \wedge F_A \rangle$. This exact form

$$(2\pi)^{-2} d\mathrm{CS}(A) \in \Omega^4(P)$$

represents the pullback of a cohomology class in $H^4(M; \mathbb{Z})$ given by

$$(2\pi)^{-2} d\mathrm{CS}(A) = -(2\pi)^{-2} \langle F_A \wedge F_A \rangle = \begin{cases} c_2 - \frac{1}{2} c_1 & \text{for } \mathrm{U}(n), \\ -\frac{1}{4} p_1 & \text{for } \mathrm{O}(n). \end{cases}$$

Compare with p. 42 and p. 164 of [DK97], given our normalization (A.18).

On a four-manifold, we define the instanton number

$$k := -(2\pi)^{-2} \int_X \langle F_A \wedge F_A \rangle = (2\pi)^{-2} \int_X d\mathrm{CS}(A). \tag{C.1}$$

To obtain a basic form, we define the Chern-Simons cocycle $\mathrm{CS} : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ by

$$\mathrm{CS}(A, A_0) := \langle a \wedge (\tfrac{1}{6} [a \wedge a] - F_A - F_{A_0}) \rangle, \text{ where } a = A - A_0. \tag{C.2}$$

In the special case of a trivial bundle, $\mathrm{CS}(A) = \mathrm{CS}(A, 0)$.

The Chern-Simons cocycle satisfies the cocycle condition

$$\mathrm{CS}(A_1, A_2) + \mathrm{CS}(A_2, A_3) + \mathrm{CS}(A_3, A_1) \in d(\Omega^2(X)).$$

Specifically,

$$\begin{aligned}
&\mathrm{CS}(A_1, A_2) + \mathrm{CS}(A_2, A_3) + \mathrm{CS}(A_3, A_1) = \\
&\qquad = \tfrac{1}{3} d \left\langle (A_1 - A_2) \wedge (A_2 - A_3) + (A_2 - A_3) \wedge (A_3 - A_1) + (A_3 - A_1) \wedge (A_1 - A_2) \right\rangle .
\end{aligned}$$

The cocycle condition implies that over a three-manifold Y, by fixing any A_0, the function

$$A \mapsto \int_Y \mathrm{CS}(A, A_0)$$

gives a well-defined map $\mathcal{A} \to \mathbb{R}$, unique up to a constant depending on A_0. Furthermore, in terms of A_0 we get the formula $\int_Y \mathrm{CS}(A_1, A_2) = \int_Y \mathrm{CS}(A_1, A_0) - \int_Y \mathrm{CS}(A_2, A_0)$.

We compute

$$\begin{aligned}
a &:= A - A_0, \\
d_A a &= F_A - F_{A_0} + \tfrac{1}{2} [a \wedge a] , \\
d_A F_{A_0} &= [a \wedge F_{A_0}] ,
\end{aligned}$$

so

$$\begin{aligned}
d\mathrm{CS}(A, A_0) &= \langle d_A a \wedge (\tfrac{1}{6} [a \wedge a] - F_A - F_{A_0}) + a \wedge (d_A F_{A_0} + \tfrac{1}{3} [a \wedge d_A a]) \rangle \\
&= \Big\langle (F_A - F_{A_0} + \tfrac{1}{2} [a \wedge a]) \wedge (\tfrac{1}{6} [a \wedge a] - F_A - F_{A_0}) + \\
&\qquad + a \wedge ([a \wedge F_{A_0}] + \tfrac{1}{3} [a \wedge (F_A - F_{A_0} + \tfrac{1}{2} [a \wedge a])]) \Big\rangle \\
&= \langle F_{A_0} \wedge F_{A_0} - F_A \wedge F_A + \tfrac{1}{4} a \wedge \cancel{[a \wedge [a \wedge a]]} \rangle \\
&= - \langle F_A \wedge F_A \rangle + \langle F_{A_0} \wedge F_{A_0} \rangle .
\end{aligned}$$

Next we will determine the effect of a gauge transformation g by computing $\mathrm{CS}(\mathrm{g}\{A\}, A)$. We will need another cocycle

$$\Delta(\mathrm{g}, A, A_0) := \Big\langle (A - A_0) \wedge (\mathrm{g}^{-1}(A - A_0)\mathrm{g} + \mathrm{g}^{-1} d_{A_0} \mathrm{g} + (d_{A_0} \mathrm{g})\mathrm{g}^{-1} \Big\rangle ,$$

which satisfies the cocycle condition

$$\Delta(\mathrm{g}, A_1, A_2) + \Delta(\mathrm{g}, A_2, A_3) + \Delta(\mathrm{g}, A_3, A_1) = 0.$$

A lengthy computation gives the identity

$$\begin{aligned}
\mathrm{CS}(\mathrm{g}\{A\}, A) &= d\Delta(\mathrm{g}, A, A_0) - \Big\langle (\mathrm{g}\{A_0\} - A_0) \wedge \big(F_{A_0} + \mathrm{g} F_{A_0} \mathrm{g}^{-1}\big) \Big\rangle + \\
&\qquad - \tfrac{1}{6} \Big\langle (\mathrm{g}^{-1} d_{A_0} \mathrm{g}) \wedge \big[(\mathrm{g}^{-1} d_{A_0} \mathrm{g}) \wedge (\mathrm{g}^{-1} d_{A_0} \mathrm{g}) \big] \Big\rangle .
\end{aligned}$$

There are several observations to be made about this identity. First, the left side is manifestly independent of A_0. Next, the term on the right side involving A is exact. Therefore, $\int_Y \mathrm{CS}(\mathrm{g}\{A\}, A)$ depends only on g. Finally, in the case $A_0 = 0$ the identity reduces to

$$\int_Y \mathrm{CS}(\mathrm{g}\{A\}, A) = -\tfrac{1}{6} \int_Y \left\langle (\mathrm{g}^{-1} d\mathrm{g}) \wedge \left[(\mathrm{g}^{-1} d\mathrm{g}) \wedge (\mathrm{g}^{-1} d\mathrm{g}) \right] \right\rangle .$$

$$\begin{aligned}
\delta \mathrm{CS}(A) &= \langle \delta A \wedge (\tfrac{1}{6} [A \wedge A] - F_A) + A \wedge (\tfrac{1}{3} [A \wedge \delta A] - d_A \delta A) \rangle \\
&= \langle \delta A \wedge (\tfrac{1}{6} [A \wedge A] + \tfrac{1}{3} [A \wedge A] - F_A) \rangle + d \langle A \wedge \delta A \rangle - 2 \langle (F_A + \tfrac{1}{2} [A \wedge A]) \wedge \delta A \rangle \\
&= d \langle A \wedge \delta A \rangle + 2 \langle \delta A \wedge -F_A \rangle .
\end{aligned}$$

In particular, on a closed 3-manifold Y,

$$\frac{\delta}{\delta A} \int_Y \mathrm{CS}(A) = -2 \star F_A .$$

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