0.0.1 Quantize the harmonic oscillator

Recall Hooke's law

$$F = -kx = -\partial_x \left(\tfrac{1}{2}kx^2\right).$$

The Lagrangian is

$$L = \tfrac{1}{2}mv^2 - \tfrac{1}{2}kx^2.$$

The Legendre transform gives

$$H = pv - L = \frac{p^2}{2m} + \tfrac{1}{2}kx^2.$$

We define

$$\omega := \sqrt{k/m}$$

so that

$$H = \tfrac{1}{2}m\left(\left(\frac{p}{m}\right)^2 + (\omega x)^2\right).$$

The Hamiltonian equations are

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial p} = \frac{p}{m},\\ \dot{p} &= -\frac{\partial H}{\partial x} = -m\omega^2 x.\end{aligned}$$

so that the classical solution is

$$\begin{aligned}x(t) &= \quad A\sin(\omega t + \theta),\\ p(t) &= m\omega A\cos(\omega t + \theta).\end{aligned}$$

The classical ground state is where energy is minimum: $x = 0$, $p = 0$, with $H = 0$.

To quantize, we replace p and x with operators

$$\begin{aligned}P &= -i\hbar\nabla,\\ X &= \text{multiplication by the coordinate function } x.\end{aligned}$$

Our Hamiltonian becomes

$$H = \tfrac{1}{2}m\left(-\left(\frac{\hbar}{m}\right)^2\nabla^2 + \omega^2 X^2\right).$$

0.0.2 Schrodinger equation and interpretation

The Schrodinger equation for the wavefunction $\psi(x,t)$ is

$$i\hbar\partial_t\psi = H\psi.$$

To solve this equation, we find eigenfunctions $\psi_E(x)$ of H with eigenvalue E. Then

$$\psi(x,t) = \exp(Et/i\hbar)\psi_E(x)$$

is a solution. Note that H is a self-adjoint operator, so under appropriate assumptions, we can find an orthonormal basis of such eigenvectors. Assume we are given some initial wavefunction $\psi(x)$ normalized so that $\|\psi\|_{L^2} = 1$. We define the coefficients

$$a_E := \int \bar{\psi}_E(x)\psi(x)\,dx.$$

The solution to the initial value problem is

$$\psi(x,t) = \sum_E a_E \exp(Et/i\hbar)\psi_E(x).$$

Given an operator $\mathbf{O}$, the expectation value of $\mathbf{O}$ on the (normalized) state ψ is denoted $\langle \mathbf{O} \rangle_\psi$ and is defined to be

$$\langle \mathbf{O} \rangle_\psi := \int \bar{\psi}(x) \mathbf{O} \psi(x)\, dx.$$

For example, let δ_x denote the Dirac delta distribution about a point x, and let $\boldsymbol{\delta}_x$ denote the corresponding multiplication operator. In this example,

$$\langle \boldsymbol{\delta}_x \rangle_\psi = |\psi(x)|^2$$

is the standard expression for the probability density for a particle to be at the position x. All quantum measurements can be expressed as expectation values. For example, the statistical variance of position is given by

$$\left\langle (X - \langle X \rangle)^2 \right\rangle_\psi.$$

0.0.3 Schrodinger equation for the harmonic oscillator

To solve the Schrodinger equation, we want to find L^2 solutions to the eigenvalue problem $H\psi = E\psi$. These correspond to elements of

$$\ker(H - E) \cap L^2.$$

Such solutions exist only for discrete values of E which turn out to be

$$E_n = \hbar\omega(n + \tfrac{1}{2}), \quad n = 0, 1, 2, \ldots$$

The minimum energy is $\frac{1}{2}\hbar\omega$ with wavefunction

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(\frac{-m\omega x^2}{2\hbar}\right). \tag{1}$$

Note that this is is a solution to the equation

$$\tfrac{1}{2} m \left(-\left(\frac{\hbar}{m}\right)^2 \psi_0''(x) + \omega^2 x^2 \psi(x) \right) = \tfrac{1}{2}\hbar\omega\psi(x)$$

with $\|\psi\|_{L^2} = 1$.

We can algebraically solve the harmonic oscillator eigenvalue problem by introducing raising and lowering operators as in Lie algebra theory:

$$\begin{aligned} a &:= \sqrt{\frac{m\omega}{2\hbar}} \left(X - \frac{P}{m\omega i} \right), \\ a^\dagger &:= \sqrt{\frac{m\omega}{2\hbar}} \left(X + \frac{P}{m\omega i} \right). \end{aligned} \tag{2}$$

The Hamiltonian can be rewritten as

$$H = \hbar\omega(a^\dagger a + \tfrac{1}{2}).$$

This choice of variables satisfies the helpful commutation relations

$$\begin{aligned} \left[a, a^\dagger\right] &= 1, \\ [H, a] &= -\hbar\omega a, \\ \left[H, a^\dagger\right] &= +\hbar\omega a. \end{aligned}$$

Consequently, if ψ is an eigenvector with eigenvalue E, then $a^\dagger\psi$ has eigenvalue $E + \hbar\omega$, and $a\psi$ has eigenvalue $E - \hbar\omega$. This would seem to imply that energy is not bounded from below. However, the descent ceases if

$$a\psi = 0.$$

Ignoring constants for the moment, this is the linear differential equation

$$(x - \partial_x)\psi = 0$$

which is solved by the Gaussian

$$\psi(x) = e^{-\frac{x^2}{2}}.$$

Keeping track of the constants, we can solve $a\psi = 0$ to derive the expression (1) for the ground state ψ_0.

The probability distribution for the position of the ground state

$$p(x) = \langle \boldsymbol{\delta}_x \rangle_{\psi_0} \tag{3}$$

is a normal distribution with standard deviation

$$\sigma = \sqrt{\frac{\hbar}{2m\omega}}.$$

0.0.4 The wave equation for a vibrating string

Consider a string of length ℓ which is fixed at two endpoints, for instance a guitar string. Let $\phi(y,t)$ denote the displacement of the string at position $y \in [0,\ell]$ and at time t. The motion is governed by the equation

$$\partial_t^2 \phi = c^2 \nabla^2 \phi,$$

for some constant c which represents the propagation speed.

The general solution is a superposition of "vibrational modes"

$$\phi(y,t) = \sum_{n=1}^{\infty} A_n \sin(\omega_n t + \theta_n) \sin(n\pi y/\ell),$$

where

$$\omega_n := \frac{c\pi}{\ell} n,$$

while A_n and θ_n are arbitrary.

Key Observation: Fourier coefficient of $\sin(n\pi y/\ell)$ is a simple harmonic oscillator of frequency ω_n.

In particular, if we denote these Fourier coefficients by $\{b_n\}$, then

$$\ddot{b}_n = -\omega_n^2 b_n.$$

The position of the string is

$$\phi(y) = \sum_{n=1}^{\infty} c_n \sin(n\pi y/\ell). \tag{4}$$

Consequently, the string is isomorphic to an infinite sequence of independent harmonic oscillators with angular frequencies

$$\{\omega_n = \omega_1 n\}_{n=1}^{\infty}.$$

0.0.5 Ground state of the quantum string

We showed in the previous section how each Fourier coefficient of a string is the position of a harmonic oscillator. Now we consider what this means for the ground state of the quantum string.

In the quantum world, the position of a harmonic oscillator in the ground state is given by a Gaussian probability distribution (3) with

$$\sigma = \sqrt{\frac{\hbar}{2m\omega}}.$$

Therefore, the coefficient c_n in our Fourier series should assume a Gaussian probability distribution with standard deviation

$$\sigma_n = \sqrt{\frac{\hbar}{2m\omega_n}} = \sqrt{\frac{\hbar\ell}{2\pi mcn}} = \frac{\sigma_1}{\sqrt{n}}.$$

Thus the ground state appears as a “random Fourier series.” To understand what such a function looks like, we recall some Fourier analysis.

Parseval's theorem states that

$$\|\phi\|_{L^2}^2 = \tfrac{1}{2}\sum c_n^2.$$

We define the Sobolev norm H^s by

$$\|\phi\|_{H^s}^2 := \left\|(\partial_x)^s \phi\right\|_{L^2}^2 = \tfrac{1}{2}\sum \left(\frac{n\pi}{\ell}\right)^{2s} c_n^2.$$

The rightmost expression makes sense for any real value of s.

Recall that variance is the square of standard deviation. For a quantity with zero mean, variance is the expected value of the square. The expected value of $\|\phi\|_{H^s}^2$ is thus

$$\tfrac{1}{2}\sum \left(\frac{n\pi}{\ell}\right)^{2s} \sigma_n^2 = \tfrac{1}{2}\left(\frac{\pi}{\ell}\right)^{2s} \sum \frac{\sigma_1^2}{n^{1-2s}},$$

which is finite only when $s < 0$. Hence ϕ is never expected to be in L^2.

The Sobolev embedding theorem, for dimension one, states that all functions in H^s are continuous whenever $s > \frac{1}{2}$. Since $\phi \in H^{-\epsilon}$ for all $\epsilon > 0$, the indefinite integral $\int \phi \in H^{1-\epsilon}$. Since $1 - \epsilon > \frac{1}{2}$ for small ϵ, we have that $\int \phi$ is continuous. In particular, we expect to get a finite number if we average ϕ over some finite interval. However, it turns out that as the width of this interval approaches zero, the corresponding averages diverge.

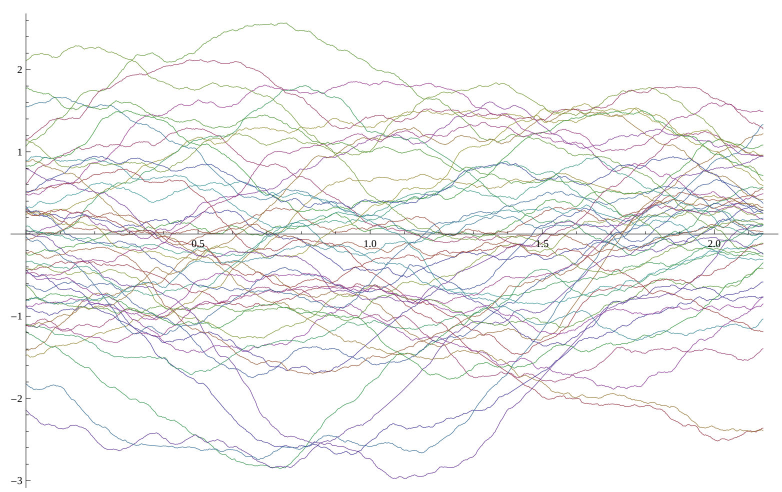

Figure 1: Fifty samples of a string of length π, averaged over intervals of length one

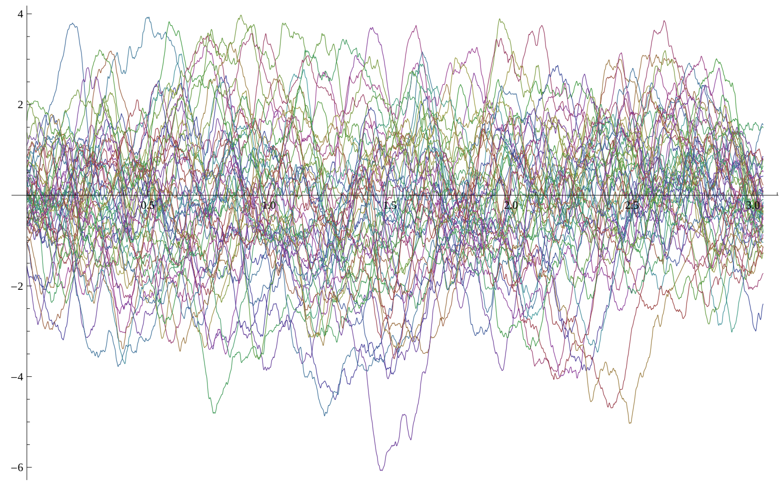

Figure 2: Fifty samples of a string of length π, averaged over intervals of length one-tenth

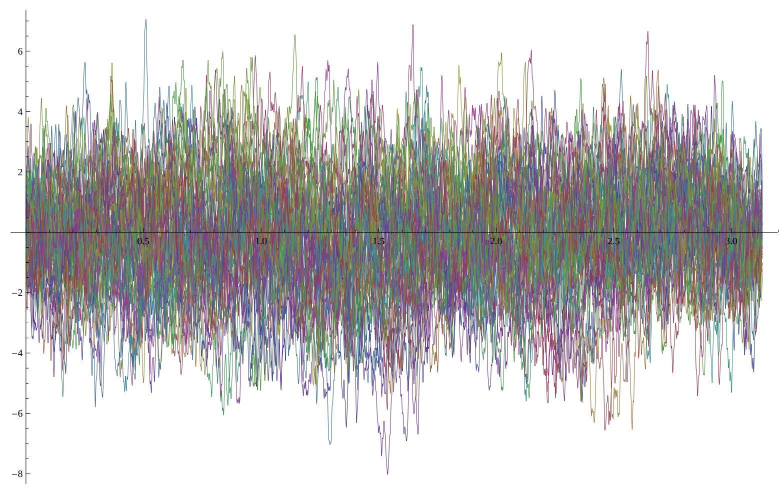

Figure 3: Fifty samples of a string of length π, averaged over intervals of length one-hundredth

0.0.6 Two-point correlation function for the quantum string

Recall that the displacement of a classical spring at the point y is given by (4)

$$\phi(y) = \sum_{n=1}^{\infty} c_n \sin(n\pi y/\ell).$$

Because $\phi(y)$ represents an observable, we should promote this to an operator $\Phi(y)$ for the quantum world. How are we to define this operator? The Fourier coefficient c_n represents the position of the n-th harmonic oscillator, so we should replace c_n with the position operator X_n. Thus we define

$$\Phi(y) := \sum_{n=1}^{\infty} X_n \sin(n\pi y/\ell).$$

The most fundamental quantitative question we can ask is how do the values at two points corellate? In other words, what's the expectation value

$$\langle \Phi(y_1)\Phi(y_2) \rangle .$$

We could tackle this problem computationally by picking a random Fourier series, evaluating it at y_1 and y_2, and multiplying the result. We would repeat for several trials and take the average of the results. Luckily there is an easier way.

$$\langle \Phi(y_1)\Phi(y_2)\rangle = \sum_{m,n=1}^{\infty} \langle X_m X_n\rangle \sin(m\pi y_1/\ell)\sin(n\pi y_2/\ell).$$

If $m \neq n$, then $\langle X_m X_n\rangle = \langle X_m\rangle\langle X_n\rangle$ since the oscillators are independent. The mean positions $\langle X_m\rangle$ and $\langle X_n\rangle$ both vanish, so the only nonzero terms are

$$\begin{aligned}\langle \Phi(y_1)\Phi(y_2)\rangle &= \sum_{n=1}^{\infty} \left\langle X_n^2\right\rangle \sin(n\pi y_1/\ell)\sin(n\pi y_2/\ell)\\ &= \sum_{n=1}^{\infty} \sigma_n^2 \sin(n\pi y_1/\ell)\sin(n\pi y_2/\ell)\\ &= \sigma_1^2 \sum_{n=1}^{\infty} \frac{\sin(n\pi y_1/\ell)\sin(n\pi y_2/\ell)}{n}.\end{aligned}$$

Note that this sum is like an alternating harmonic series, and is only conditionally convergent! Physically, for this to be sensible, we should think of n ranging up to some very large but finite cutoff.

We can explicitly evaluate our sum in terms of complex exponentials, making use of the Maclaurin series for $\ln(1+x)$:

$$\sum_{n=1}^{\infty} \frac{e^{inx}}{n} = -\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-e^{ix})^n}{n} = -\ln(1-e^{ix}).$$

The final answer is

$$\langle \Phi(y_1)\Phi(y_2)\rangle = \frac{\hbar\ell}{8\pi mc}\ln\left(\frac{1-\cos\left(\frac{\pi}{\ell}(y_1+y_2)\right)}{1-\cos\left(\frac{\pi}{\ell}(y_1-y_2)\right)}\right).$$

The asymptote at $y_1 = y_2$ indicates that the expected "pointwise value squared" $\left\langle \Phi(y)^2\right\rangle$ is divergent.

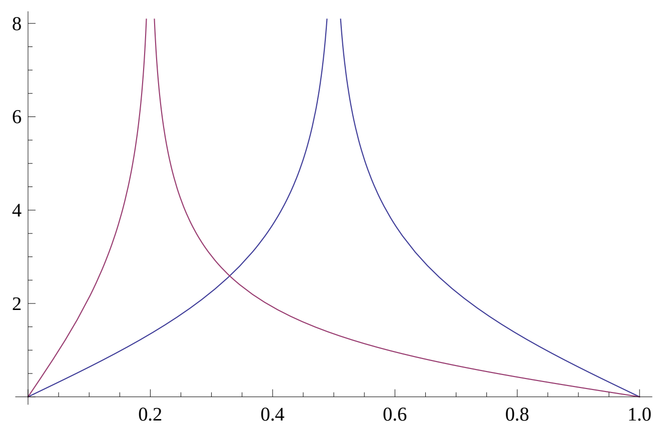

Figure 4: The two-point correlation function for a quantum string of unit length. The horizontal axis is y_1, and the two curves represent $y_2 = \frac{1}{2}$ and $y_2 = \frac{1}{4}$. The vertical axis is in units of $\frac{\hbar\ell}{8\pi mc}$.