Visualizing orientations and differential forms

Orientations

Suppose $\omega \in \Omega^n(M^n)$, and ω is nowhere zero. Then for each $p \in M$, $\omega|_p \in \Lambda^n T_p^* M$, and $\omega|_p$ is nowhere zero.

What can we do with $\omega|_p$? We can evaluate it on an n-tuple of tangent vectors in T_pM. Evaluation gives us the correspondence between

$$\Lambda^n T_p^* M \longleftrightarrow \mathrm{Alt}^n T_pM,$$

so we get a nonzero alternating n-linear map $\omega|_p$ with

$$\omega|_p(v_1, \ldots, v_n) \in \mathbb{R}.$$

Recall that $\mathrm{Alt}^n \mathbb{R}^n$ is $\binom{n}{n} = 1$-dimensional, spanned by the determinant. Thus if we were to choose a basis of T_pM, then $\omega|_p$ would be proportional to the determinant. The key property of the determinant is that it gives the signed volume of the parallelepiped with edges given by column vectors $(v_1, \ldots, v_n)$. In particular, the determinant is zero iff the columns are linearly dependent.

It's bad to choose a basis unless absolutely necessary, so let's think of T_pM as an abstract n-dimensional vector space. Then $\omega|_p$ corresponds to some alternating function on T_pM, which behaves like a nonzero multiple of the determinant. Abstractly, this gives us a notion of signed volume in T_pM. Furthermore,

$$\omega|_p(v_1, \ldots, v_n) \neq 0 \iff \{v_1, \ldots, v_n\} \text{ linearly independent } \iff \{v_1, \ldots, v_n\} \text{ form a basis for } T_pM.$$

In other words,

Given any ordered basis $(v_1, \ldots, v_n)$ of T_pM, then $\omega|_p(v_1, \ldots, v_n)$ is either positive or negative.

Any ordered basis $(v_1, \ldots, v_n)$ of T_pM is said to be either a *positive basis* or a *negative basis* according to the sign of $\omega|_p(v_1, \ldots, v_n)$.

Recall that two orientation forms $\omega_1, \omega_2 \in \Omega^n(M)$ are *equivalent* if they are multiples of each other by positive functions $\omega_2 = f\omega_1$, $\omega_1 = (1/f)\omega_2$, with $f(p) > 0 \; \forall p \in M$. In this case,

$$\omega|_p(v_1, \ldots, v_n) \text{ is positive } \iff f(p)\, \omega|_p(v_1, \ldots, v_n) \text{ is positive.}$$

Thus,

An orientation determined by $\omega \in \Omega^n(M^n)$ gives a continuously varying notion of which bases are positive vs. negative.

Indeed, any continuously varying notion of positive vs. negative basis comes from an orientation, so the two notions are equivalent.

Positive and negative ordered bases

Any ordered basis $(v_1, \ldots, v_n)$ of T_pM is either positively or negatively oriented according to the value of $\omega|_p$. The value of $\omega|_p(v_1, \ldots, v_n)$ varies *continuously* with the v_i, just as the determinant function is continuous. Consequently, if I consider a homotopy of the basis vectors, making sure that they remain a basis for all time values, then it's impossible for the basis to switch sign. (Suppose my basis at $t = 0$ is positive while my basis at $t = 1$ basis is negative. Then by the intermediate value theorem, there is some $t_0 \in (0, 1)$ where the basis evaluates to zero. This is impossible since every basis is either positive or negative.)

To illustrate, suppose we have some ordered basis (v_r, v_g, v_b) of T_pM.

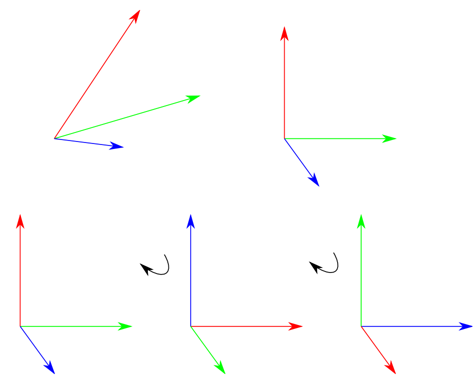

Figure 1: All these ordered bases have the same sign.

Without changing the sign, we can homotope (v_r, v_g, v_b) into some "standard position" and use rotations to cyclically permute them.

It turns out that using rotations, we can only achieve even permutations (i.e. permutations $\sigma \in S_n$ with $\text{sign}(\sigma) = +1$). We can get the odd permutations with a reflection.

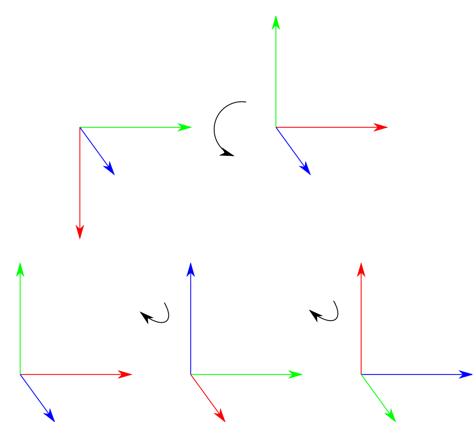

Figure 2: All these ordered bases have the opposite sign to those in Figure 1 on page 2.

We know a priori that the ordered bases in Figure 1 on page 2 and Figure 2 on page 2 have opposite signs. The role of $\omega|_p$ is to distinguish which is positive and which is negative.