Suppose $(M^n, [\omega_0])$ is an oriented manifold, and $\omega \in \Omega^n(M)$. Last time we assumed convergence of both $\int_{U_\alpha}$ and $\sum_\alpha$, and showed that

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} \phi_\alpha \omega \in \mathbb{R}$$

is well-defined, where ϕ_α is any partition of unity subordinate to a positive atlas for the orientation $[\omega_0]$.

How do we ensure covergence?

Compact supports

$$\Omega^p_c(M) := \{\omega \in \Omega^p(M) \,|\, \mathrm{supp}(\omega) \text{ is compact}\}.$$

We will show that

$$\int_{M^n} \; : \; \Omega^n_c \to \mathbb{R}$$

is a well-defined linear map.

Note: Any closed subset of a compact set is compact.

Example 1. If M is compact, then for any $\omega \in \Omega^p(M)$, $\mathrm{supp}(\omega) \subset M$ is by definition closed (recall that support is the *closure* of the set where ω is nonzero), hence compact. Thus

$$\Omega^p(M) = \Omega^p_c(M) \text{ if } M \text{ is compact.}$$

Example 2. Think of the case $M = \mathring{D}^2$, the open two-ball. For ω to have compact support, the support must stay away from the boundary. For manifolds which arise as subsets of $\mathbb{R}^n$, we use the relative topology. Thus the closure of a subset of M can't contain points outside of M. If the support extends along the edge of M, its relative closure will not be closed as a subset of $\mathbb{R}^2$, hence noncompact.

Example 3. Consider an infinite cylinder. ω is compact iff it is nonzero outside a set of finite width.

Convergence of $\int_{U_\alpha} \phi_\alpha \omega$ for $\omega \in \Omega^n_c(M)$

Note that $\mathrm{supp}(\phi_\alpha \omega) \subset \mathrm{supp}(\omega)$ is a closed subset of a compact set. Thus $\mathrm{supp}(\phi_\alpha \omega)$ is compact.

If we implicitly identify $U_\alpha \subset M$ with its image $h_\alpha(U_\alpha) \subset \mathbb{R}^n$, then $\phi_\alpha \omega$ extends by zero to $\mathbb{R}^n$, and

$$\int_{U_\alpha} \phi_\alpha \omega = \int_{\mathbb{R}^n} \phi_\alpha \omega.$$

In coordinates, we can write $\phi_\alpha \omega = f_\alpha \, dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha$, where $\mathrm{supp}(f_\alpha) = \mathrm{supp}(\phi_\alpha \omega)$ which is compact, and

$$\int_{U_\alpha} \phi_\alpha \omega = \int_{\mathbb{R}^n} f_\alpha \, dx_1^\alpha \cdots dx_n^\alpha.$$

Since compact subsets of $\mathbb{R}^n$ are closed and bounded, we know that f_α is nonzero on a bounded subset of $\mathbb{R}^n$. Also, by the extreme value theorem, since f_α is a continuous function with compact support, its values are bounded. Finally, since f_α is a continuous and bounded function which is nonzero on a bounded set, it is Riemann integrable. Thus

$$\int_{\mathbb{R}^n} f_\alpha \, dx_1^\alpha \cdots dx_n^\alpha \in \mathbb{R}$$

is well-defined, so $\int_{U_\alpha} \phi_\alpha \omega$ converges.

Convergence of $\sum_\alpha$

It turns out that by the local finiteness property for a partition of unity, the sum in

$$\sum_\alpha \int_{U_\alpha} \phi_\alpha \omega$$

has only *finitely many* nonzero terms when $\omega \in \Omega_c^n(M)$.

Recall that a partition of unity $\{\phi_\alpha\}$ is *locally finite,* which means that each $p \in M$ is contained in some open $U_p \in M$ along which only finitely many $\phi_\alpha|_{U_p}$ are nonzero.

Suppose that $\omega \in \Omega_c^n(M)$. Cover $\operatorname{supp}(\omega)$ by $\{U_p | p \in \operatorname{supp}(\omega)\}$. Since $\operatorname{supp}(\omega)$ is compact, there is some finite subcover $U_1, \ldots, U_s$ for $\operatorname{supp}(\omega)$. Each of these U_i has only finitely many nonzero ϕ_α. There are only finitely many U_i which cover $\operatorname{supp}(\omega)$. Thus there are only finitely many nonzero ϕ_α on $\operatorname{supp}(\omega)$. Thus $\sum_\alpha$ is finite, so there are no convergence issues.

We have shown that $\sum_\alpha \int_{U_\alpha} \phi_\alpha \omega$ always converges. Last time we showed that it always converges to the same thing, i.e. it is well-defined. Thus we have achieved our goal of showing that

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} \phi_\alpha \omega \in \mathbb{R}$$

is well-defined for $\omega \in \Omega_c^n(M)$.

Note: What happens if we use the opposite orientation $[-\omega_0]$ on M? To get a positive atlas corresponding to $[-\omega_0]$, we must reflect the coordinate charts for the positive atlas corresponding to $[\omega_0]$. In the reflected coordinate charts, the reflection introduces a factor of -1 for each coefficient functions f_α. Thus

$$\int_{(M,[-\omega_0])} \omega = -\int_{(M,[\omega_0])} \omega.$$

Stokes' Theorem (version 1)

We now prove our first version of Stokes' Theorem.

Theorem. *If M^n is an oriented manifold, and $\eta \in \Omega_c^{n-1}(M)$, then*

$$\int_M d\eta = 0.$$

Proof. We write

$$\eta = \sum_\alpha \phi_\alpha \eta$$

so that

$$\int_M d\eta = \sum_\alpha \int_M d(\phi_\alpha \eta).$$

(Note that this is not a direct application of the definition of $\int_M d\eta$. That would instead give us $\sum_\alpha \int_M \phi_\alpha d\eta$, which is not suitable for this proof.)

Note that applying d can only decrease the support of a differential form. (A differential form is identically zero outside the support, and applying d won't change this.) We know that $\operatorname{supp}(\phi_\alpha \eta) \subset U_\alpha$, so $\operatorname{supp}(d(\phi_\alpha \eta)) \subset U_\alpha$ as well. Thus

$$\int_M d\eta = \sum_\alpha \int_{U_\alpha} d(\phi_\alpha \eta) = \sum_\alpha \int_{\mathbb{R}^n} d(\phi_\alpha \eta).$$

If we can show that

$$\int_{\mathbb{R}^n} d(\phi_\alpha \eta) = 0$$

for each α, then we have the theorem. By replacing $\phi_\alpha \eta$ with a general element $\eta \in \Omega_c^{n-1}(\mathbb{R}^n)$, it suffices to show that

$$\int_{\mathbb{R}^n} d\eta = 0.$$

Thus we have reduced the computation to a computation in $\mathbb{R}^n$.

We write out η in coordinates as

$$\begin{aligned}
\eta &= \eta_1\, dx_2 \wedge \cdots \wedge dx_n - \eta_2\, dx_1 \wedge dx_3 \wedge dx_4 \wedge \cdots \wedge dx_n + \cdots \\
&= \sum_i (-1)^{i-1} \eta_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,
\end{aligned}$$

where $\widehat{dx_i}$ denotes that dx_i is omitted. The reason for the factor of $(-1)^{i-1}$ becomes clear when we compute

$$\begin{aligned}
d\eta &= \sum_i (-1)^{i-1} \frac{\partial \eta_i}{\partial x_i} dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&= \left(\frac{\partial \eta_1}{\partial x_1} + \cdots + \frac{\partial \eta_n}{\partial x_n} \right) dx_1 \wedge \cdots \wedge dx_n.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{\mathbb{R}^n} d\eta &= \int_{\mathbb{R}^n} \left(\frac{\partial \eta_1}{\partial x_1} + \cdots + \frac{\partial \eta_n}{\partial x_n} \right) dx_1 \cdots dx_n \\
&= \sum_i \int_{\mathbb{R}^n} \frac{\partial \eta_i}{\partial x_i} dx_1 \cdots dx_n \\
&= \sum_i \int_{\mathbb{R}^{n-1}} \left[\int_{-\infty}^{\infty} \frac{\partial \eta_i}{\partial x_i} dx_i \right] dx_1 \cdots \widehat{dx_i} \cdots dx_n.
\end{aligned}$$

It suffices to show that

$$\int_{-\infty}^{\infty} \frac{\partial \eta_i}{\partial x_i} dx_i = 0.$$

By the Fundamental Theorem of Calculus, the definite integral is

$$\int_a^b \frac{\partial \eta_i}{\partial x_i} dx_i = \eta_i(x_1, \ldots, \underbrace{b}_{x_i}, \ldots, x_n) - \eta_i(x_1, \ldots, \underbrace{a}_{x_i}, \ldots, x_n).$$

Now recall that η has compact support. It vanishes outside of a bounded set. In particular, if *any* single coordinate becomes too large, then $\eta_i = 0$. In particular, when b is sufficiently large positive, and when a is sufficiently large negative, we have

$$\int_a^b \frac{\partial \eta_i}{\partial x_i} dx_i = 0 - 0.$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{\partial \eta_i}{\partial x_i} dx_i = 0.$$

Unwinding the reductions we made, this shows that

$$0 = \sum_i \int_{\mathbb{R}^{n-1}} \left[\int_{-\infty}^{\infty} \frac{\partial \eta_i}{\partial x_i} dx_i \right] dx_1 \cdots \widehat{dx_i} \cdots dx_n,$$

so

$$0 = \int_{\mathbb{R}^n} d\eta,$$

and

$$0 = \int_{U_\alpha} d(\phi_\alpha \eta),$$

and finally

$$0 = \int_M d\eta.$$

□

Next time we will introduce the notion of manifolds with boundary, and we will prove a more general version of Stokes' Theorem:

$$\int_{\partial M} \eta = \int_M d\eta.$$