Orientations and linear maps

The manifold $\mathbb{R}^n$ has two possible orientations, determined by

$$\pm dx_1 \wedge \cdots \wedge dx_n.$$

Let's wean ourselves away from using normal vectors to visualize orientations.

We begin with the well-known fact from linear algebra that every invertible square matrix can be written as a product of elementary matrices `http://en.wikipedia.org/wiki/Elementary_matrix`. Correspondingly, every invertible transformation of $\mathbb{R}^n$ is a composition of the following transformations:

- Coordinate swap $x_i \leftrightarrow x_j$,
- Scale transformation $x_i \to kx_i$ with $k \neq 0$,
- Skew transformation $x_i \to x_i + kx_j$ with $i \neq j$.

Note that a coordinate swap corresponds to reflection about the plane $x_i = x_j$. Also, a scale transformation with $k = -1$ corresponds to a reflection about the $x_i = 0$ plane. After possibly composing with this $k = -1$ reflection, we can assume that $k > 0$. Thus we consider four types of transformations:

- Coordinate swap $x_i \leftrightarrow x_j$,
- Scale transformation $x_i \to -x_i$,
- Scale transformation $x_i \to kx_i$ with $k > 0$,
- Skew transformation $x_i \to x_i + kx_j$ with $i \neq j$.

Problem 1. How do each of these four elementary transformations affect the two orientations of $\mathbb{R}^n$? How do the first two "reflective" transformations compare with the last two "non-reflective" transformations?

In your head, think about the properties you would expect from a sensible notion of "orientation." According to what you would expect, do the above results make sense?

Cotangent vectors

Consider $p \in M^n$ and $\alpha \in T_p^*M$. Suppose that in a local coordinate chart,

$$\alpha = \alpha_1 \, dx_1|_p + \cdots + \alpha_n \, dx_n|_p$$

for some coefficients where $\alpha_i \in \mathbb{R}$.

Problem 2. Construct some $f \in \Omega^0(M)$ so that $df|_p = \alpha$.

Note: make sure f is defined on all of M, and not just within the coordinate chart.

Tuesday option

Derivations

Recall that in local coordinates, a vector field V looks like

$$V = V_1 \frac{\partial}{\partial x_1} + \cdots + V_n \frac{\partial}{\partial x_n},$$

for smooth functions V_i. This notation suggests that V acts as a directional derivative operator. Specifically, given a function $f \in \Omega^0(M)$, we define $Vf \in \Omega^0(M)$, which is given locally by

$$Vf = V_1 \frac{\partial f}{\partial x_1} + \cdots + V_n \frac{\partial f}{\partial x_n}.$$

Problem 3. Prove that V is a *derivation* on $\Omega^0(M)$, i.e.

$$V(\alpha f + \beta g) = \alpha V f + \beta V g \text{ for } \alpha, \beta \in \mathbb{R} \text{ and } f, g \in \Omega^0(M),$$
$$V(fg) = g\, Vf + f\, Vg \text{ for } f, g \in \Omega^0(M).$$

Hint: Since a function is determined by its values at each point, it suffices to check these formulas locally. There is no need to reverify the transformation rule. (In class I already verified in painstaking detail that the transformation rule checks out.)

Problem 4. Referring to your answer for Problem Set 4, Problem 1, show that in the case $M = \mathbb{R}$, every derivation determines a vector field on M. Thus for $M = \mathbb{R}$, "vector fields on $\mathbb{R}$" are equivalent to "derivations on $\Omega^0(\mathbb{R})$."

Note: it's not difficult to prove that for a general manifold M, "vector fields on M" are equivalent to "derivations on $\Omega^0(M)$." The proof is a slight modification of your answer for Problem Set 4, Problem 1.