## The nerve of an open cover

As in Example 8.5 [M&T], Consider the two-sphere $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ with the atlas of six charts $\{(U_{\pm i}, h_{\pm i})\}_{i=1}^{3}$ where the $U_{\pm i}$ are given by

$$U_{+i} = \left\{x \in S^2 \mid x_i > 0\right\},$$
$$U_{-i} = \left\{x \in S^2 \mid x_i < 0\right\}.$$

**Problem 1.** Show that there is no quadruple intersection of the $U_{\pm i}$, i.e. show that the intersection of any four distinct $U_{\pm i}$ is empty.

The nerve of a cover is constructed as follows. For every open set $U$ in our cover, we draw a vertex.

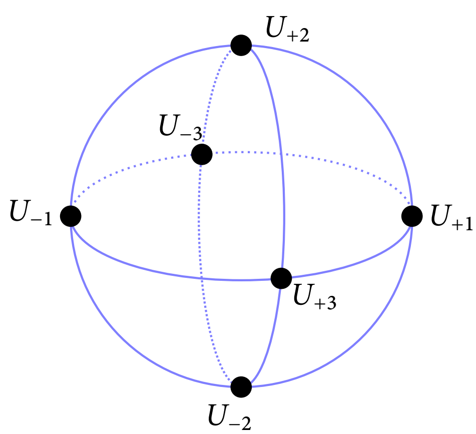
Draw a line segment for each pair of vertices for which the pairwise intersection is nonempty.

For each triple of vertices where the triple-intersection is nonempty, fill in the corresponding triangle.

For each quadruple of vertices where the quadruple-intersection is nonempty, fill in the corresponding tetrahedron.

(This construction continues to higher and higher dimensions.)

**Problem 2.** Sketch (some perspective of) the nerve of this cover $\{U_{\pm i}\}_{i=1}^{3}$, and briefly describe the shape. It will be helpful to think of each vertex as being located at the center of each open set. By Problem 7, we know that there are no tetrahedra to fill in.




*Remark.* A *good cover* for a manifold is an open cover for which all nonempty multiple-intersections are contractible. Such covers are good because the nerve encodes the topology of the manifold.

## Definition of an atlas

Consider the set $X = \mathbb{R}$, and the standard atlas $\mathcal{A}_1 = \{(U_1, h_1)\} = \{(\mathbb{R}, \mathrm{id}_{\mathbb{R}})\}$.

Consider the function cube : $\mathbb{R} \to \mathbb{R}$ given by $\mathrm{cube}(x) = x^3$.

**Problem 3.** Is $\mathcal{A}_2 = \{(\mathbb{R}, \text{cube})\}$ an atlas? Is it a smooth atlas? Why or why not?

**Problem 4.** Are $\mathcal{A}_1$ and $\mathcal{A}_2$ compatible? Why or why not?

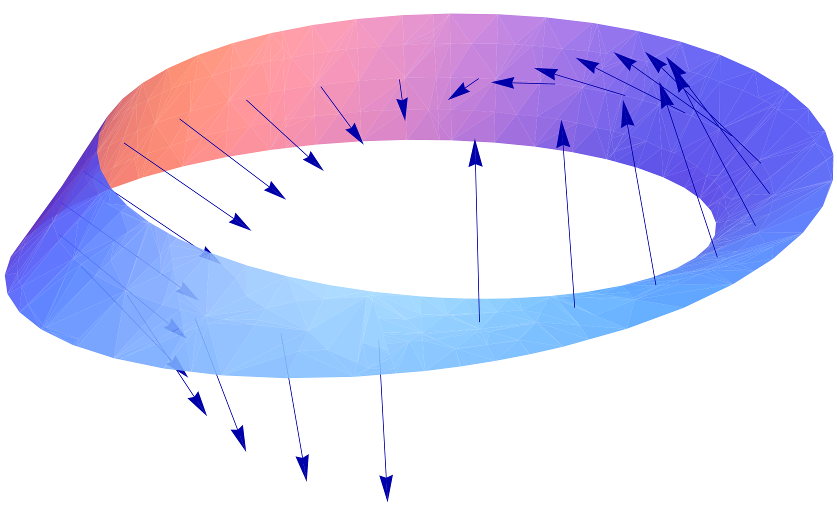
Recall that a diffeomorphism of manifolds $\phi : M_1 \to M_2$ is a smooth function with a smooth inverse.

**Problem 5.** Are the manifolds associated to $\mathcal{A}_1$ and $\mathcal{A}_2$ diffeomorphic? If so, give a diffeomorphism. If not, explain why.

Hint: In order to check that a map between manifolds is smooth, you must verify that the corresponding map under the coordinate charts is smooth. (See M&T 8.7 or Hitchin Chapter 1, 2.4.) I drew a corresponding diagram at some point in class.

## Orientations

**Definition.** A smooth manifold $M$ of dimension $n$ is *orientable* if there exists some $\omega \in \Omega^n(M)$ such that $\omega$ is nowhere zero. Such an $\omega$ is called an *orientation form*. Two orientation forms $\omega_1$ and $\omega_2$ are *equivalent* if $\omega_1 = f \cdot \omega_2$ for some everywhere-positive $f \in \Omega^0(M)$. An *orientation* on $M$ is an equivalence class of orientation forms. (M&T 9.8)



Suppose $\Sigma^2$ is a two-dimensional submanifold of $\mathbb{R}^3$, and $x \in \Sigma$. Let $\vec{n}$ be a nonzero vector which is normal to $\Sigma$ at $x$. Define a function $T_x\Sigma \times T_x\Sigma \to \mathbb{R}$ denoted by $\vec{n}$ by

$$\vec{n}(\vec{t}_1, \vec{t}_2) := \vec{n} \cdot (\vec{t}_1 \times \vec{t}_2).$$

**Problem 6.** Show that $\vec{n}(\vec{t}_1, \vec{t}_2)$ is not the zero map on $T_x\Sigma$. (Hint: complete $\vec{n}$ to an orthonormal basis.) Show that this function is alternating, so that $\vec{n}$ defines a nonzero element of $\text{Alt}^2(T_x\Sigma)$. (M&T 2.1)

Hint: $T_x\Sigma$ denotes the "tangent space" to $\Sigma$ at the point $x$. Although we haven't discussed it yet in class, for these problems you should view it as the tangent plane at $x$. (It's a vector space, where all vectors have the initial point $x$.)

Suppose further that $\vec{n}$ extends to a nowhere vanishing field of smoothly-varying normal vectors along $\Sigma$.

**Problem 7.** According to the previous problem, what does such a vector field determine on $\Sigma$? (M&T 9.5) What does this say about the orientability of $\Sigma$?