First application of cohomology

Problem 1. Let's prove

Theorem. $\mathbb{R}^2 - \{0\}$ *is not diffeomorphic to any star-shaped region.*

First show that H^p is a functor.

Hint: Suppose $\phi : U_1 \to U_2$. Then verify the following statements. (The proofs should be about two lines each.) Look up the definitions of "cocycle," "coboundary" and "well-defined" in the "Background material" notes.

- If $\omega \in Z^p(U_2)$, then $\phi^*(\omega) \in Z^p(U_1)$.
- If $\omega \in B^p(U_2)$, then $\phi^*(\omega) \in B^p(U_1)$.
- If $[\omega] \in H^p(U_2)$, then $\phi^*([\omega]) := [\phi^*(\omega)]$ is well-defined.
- If ϕ is the identity on U, then ϕ^* is the identity on $H^p(U)$.
- $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

Now that we know H^p is a functor, explain why a diffeomorphism $\phi : U_1 \to U_2$ induces an isomorphism $\phi^* : H^p(U_2) \to H^p(U_1)$. (Again, two lines.)

Now consider

$$\alpha = \frac{x_1\, dx_2 - x_2\, dx_1}{x_1^2 + x_2^2} \in \Omega^1(\mathbb{R}^2 - \{0\}).$$

Verify that $\alpha \in Z^p(\mathbb{R}^2 - \{0\})$.

Define $\gamma : [0, 2\pi] \to \mathbb{R}^2 - \{0\}$ by $\gamma(t) = (\cos t, \sin t)$. Compute $\gamma^*(\alpha)$.

Based on the suggestive notation from your answer above, compute

$$\int \gamma^*(\alpha) \in \mathbb{R}.$$

Show that

$$\alpha \mapsto \int \gamma^*(\alpha)$$

is a linear map $\Omega^1(\mathbb{R}^2 - \{0\}) \to \mathbb{R}$.

Finally, use the Fundamental Theorem of Calculus to show that

$$[\alpha] \mapsto \int \gamma^*(\alpha)$$

is a well-defined map

$$H^1(\mathbb{R}^2 - \{0\}) \to \mathbb{R}.$$

Put together the above statements to conclude the theorem.

Maxwell's equations

Maxwell's equations describe the electromagnetic field. The electric field $\mathbf{E}(t) = \mathbf{E} = (E_1, E_2, E_3)$ and magnetic field $\mathbf{B}(t) = \mathbf{B} = (B_1, B_2, B_3)$ are vector fields on $\mathbb{R}^3$. Maxwell's equations are

$$\begin{aligned}\nabla \cdot \mathbf{B} &= 0,\\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t},\\ \nabla \cdot \mathbf{E} &= 4\pi\rho,\\ \nabla \times \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{J},\end{aligned}$$

where ρ is the electric charge density and $\mathbf{J}$ is the electric current.

Spacetime is $\mathbb{R}^4$ with the coordinates (x_1, x_2, x_3, t). The electromagnetic field $F \in \Omega^2(\mathbb{R}^4)$ is

$$F := (E_1\, dx_1 + E_2\, dx_2 + E_3\, dx_3) \wedge dt + B_1\, dx_2 \wedge dx_3 + B_2\, dx_3 \wedge dx_1 + B_3\, dx_1 \wedge dx_2.$$

Problem 2. Show that

$$dF = 0 \iff \begin{cases} \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}. \end{cases}$$

Hint: Notice that the formula for F is invariant under the "cyclic permutation" $\begin{matrix} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 1 \end{matrix}$. Similarly, the component formulas for the dot and cross products have this same symmetry. You can greatly reduce your paper usage by exploiting cyclic permutations. For example,

$$\begin{aligned} F &= E_1\, dx_1 \wedge dt + B_1\, dx_2 \wedge dx_3 + \text{c.p.} \\ \nabla \times \mathbf{E} &= (\partial_2 E_3 - \partial_3 E_2)\vec{e}_1 + \text{c.p.} \end{aligned}$$

where c.p. denotes the sum over the remaining two cyclic permutations $\begin{matrix} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 1 \end{matrix}$, and $\begin{matrix} 1 \mapsto 3 \\ 2 \mapsto 1 \\ 3 \mapsto 2 \end{matrix}$.

To get the other two equations, we will need the Hodge star operator $\star$, which encodes the Minkowski structure of spacetime, and basically exchanges the electric and magnetic fields. Rather than define it, the result we need is:

$$\star F = (-B_1\, dx_1 - B_2\, dx_2 - B_3\, dx_3) \wedge dt + E_1\, dx_2 \wedge dx_3 + E_2\, dx_3 \wedge dx_1 + E_3\, dx_1 \wedge dx_2.$$

We define

$$J = \rho\, dx_1 \wedge dx_2 \wedge dx_3 - dt \wedge (J_1\, dx_2 \wedge dx_3 + J_2\, dx_3 \wedge dx_1 + J_3\, dx_1 \wedge dx_2).$$

Problem 3. Show that

$$d \star F = 4\pi J \iff \begin{cases} \nabla \cdot \mathbf{E} &= 4\pi\rho, \\ \nabla \times \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{J}. \end{cases}$$

Therefore, Maxwell's equations are equivalent to:

$$dF = 0$$
$$d \star F = 4\pi J.$$

Problem 4. Use differential forms to prove that any solution of Maxwell's equations must satisfy the "conservation of charge" law $\dot{\rho} + \operatorname{div}(\mathbf{J}) = 0$.

Hint: compute dJ in two different ways. Your solution should be short.

The Poincaré lemma

Problem 5. Explain in two or fewer sentences (with equations) precisely why the electromagnetic field $F \in \Omega^2(\mathbb{R}^4)$ is determined by an "electromagnetic potential" $A \in \Omega^1(\mathbb{R}^4)$ satisfying $F = dA$.

Problem 6. Show that for any $A \in \Omega^1(\mathbb{R}^4)$ and $f \in \Omega^0(\mathbb{R}^4)$, the electromagnetic potentials A and $A + df$ represent the same electromagnetic field. Show that if $A_1, A_2 \in \Omega^1(\mathbb{R}^4)$ represent the same electromagnetic field, then $A_2 = A_1 + df$ for some $f \in \Omega^0(\mathbb{R}^4)$.

Remark. While the electromagnetic field is straightforward to measure, the "Aharonov-Bohm effect" shows that cohomology of the electromagnetic potential is also measurable. A solenoid running along the x_3-axis (and for all times t) generates an electromagnetic potential proportional to

$$A = \frac{x_1\,dx_2 - x_2\,dx_1}{x_1^2 + x_2^2} \in \Omega^1\left(\left\{(t, x_1, x_2, x_3) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 \neq 0\right\}\right).$$

Although the corresponding electromagnetic field is everywhere zero, the nonzero integral of Problem 1 leads to an observable quantum interference pattern.

Tuesday option

The nerve of an open cover

As in Example 8.5 [M&T], Consider the two-sphere $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ with the atlas of six charts $\{(U_{\pm i}, h_{\pm i})\}_{i=1}^3$ where the $U_{\pm i}$ are given by

$$U_{+i} = \left\{x \in S^2 \mid x_i > 0\right\},$$
$$U_{-i} = \left\{x \in S^2 \mid x_i < 0\right\}.$$

Problem 7. Show that there is no quadruple intersection of the $U_{\pm i}$, i.e. show that the intersection of any four distinct $U_{\pm i}$ is empty.

The nerve of a cover is constructed as follows. For every open set U in our cover, we draw a vertex.

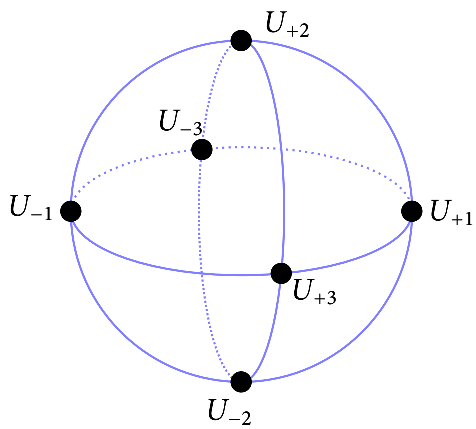
Draw a line segment for each pair of vertices for which the pairwise intersection is nonempty.

For each triple of vertices where the triple-intersection is nonempty, fill in the corresponding triangle.

For each quadruple of vertices where the quadruple-intersection is nonempty, fill in the corresponding tetrahedron.

(This construction continues to higher and higher dimensions.)

Problem 8. Sketch (some perspective of) the nerve of this cover $\{U_{\pm i}\}_{i=1}^{3}$, and briefly describe the shape. It will be helpful to think of each vertex as being located at the center of each open set. By Problem 7, we know that there are no tetrahedra to fill in.



Remark. A *good cover* for a manifold is an open cover for which all nonempty multiple-intersections are contractible. Such covers are good because the nerve encodes the topology of the manifold.