Problem 1. Consider the permutation σ given by

n	1	2	3	4	5	6
$\sigma(n)$	4	3	1	5	6	2

Compute $\operatorname{sign}(\sigma)$.

Problem 2. Compute

$$\varepsilon_4 \wedge \varepsilon_3 \wedge \varepsilon_1 \wedge \varepsilon_5 \wedge \varepsilon_6 \wedge \varepsilon_2 \left(e_1, e_2, e_3, e_4, e_5, e_6\right).$$

Problem 3. Compute

$$(\varepsilon_1 + 3\varepsilon_2 + 2\varepsilon_3) \wedge (9\varepsilon_1 + 5\varepsilon_2 + 7\varepsilon_3) \wedge (7\varepsilon_1 + 3\varepsilon_2 + 5\varepsilon_3).$$

Problem 4. Identify the components of

$$(v_1\varepsilon_1 + v_2\varepsilon_2 + v_3\varepsilon_3) \wedge (w_1\varepsilon_1 + w_2\varepsilon_2 + w_3\varepsilon_3)$$

with the components of the cross product $v \times w$.

Problem 5. For $\omega = \varepsilon_1 \wedge \varepsilon_2 + \varepsilon_3 \wedge \varepsilon_4 + \varepsilon_5 \wedge \varepsilon_6$, compute

- $\omega \wedge \omega$
- $\omega \wedge \omega \wedge \omega$
- $\omega \wedge \omega \wedge \omega \wedge \omega$

Problem (Graduate, M&T 2.10)**.** Let V be a 4-dimensional vector space and $\{\varepsilon_1, \dots, \varepsilon_4\}$ a basis for $\Lambda^1(V^*)$. Let $A = \left[a_{ij}\right]$ be a *skew-symmetric* matrix and define

$$\alpha = \sum_{i<j} a_{ij}\, \varepsilon_i \wedge \varepsilon_j.$$

Show that $\alpha \wedge \alpha = 0 \iff \det A = 0$. Say $\alpha \wedge \alpha = \lambda.\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4$. What is the relation between λ and $\det(A)$?

Hint:

$$\det \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = (af - be + cd)^2.$$

Problem 6. Consider the map $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\phi(x_1, x_2, x_3) = (x_1x_2, x_2x_3).$$

Suppose y_1, y_2 are coordinate functions on $\mathbb{R}^2$ with corresponding one-forms dy_1, dy_2. Let $\alpha \in \Omega^1(\mathbb{R}^2)$ be the one-form given by

$$\alpha = y_2 dy_1 - y_1^2 dy_2.$$

- Compute $\phi^*(\alpha) \in \Omega^1(\mathbb{R}^3)$.
- Compute $d\alpha$.
- Compute $\phi^*(d\alpha)$.
- Compute $d\phi^*(\alpha)$.

Problem 7 (M&T 3.1). Show for an open set in $\mathbb{R}^2$ that the de Rham complex

$$\Omega^0(U) \to \Omega^1(U) \to \Omega^2(U) \to 0$$

is isomorphic to the complex

$$C^\infty(U,\mathbb{R}) \xrightarrow{\vec{\nabla}} C^\infty(U,\mathbb{R}^2) \xrightarrow{\text{rot}} C^\infty(U,\mathbb{R}) \to 0.$$

Analogously, show that for an open set in $\mathbb{R}^3$ the de Rham complex is isomorphic to

$$C^\infty(U,\mathbb{R}) \xrightarrow{\vec{\nabla}} C^\infty(U,\mathbb{R}^3) \xrightarrow{\vec{\text{rot}}} C^\infty(U,\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(U,\mathbb{R}) \to 0.$$

The Tuesday option

Here is the first problem from the next problem set. If you want, you may instead hand in this assignment on Tuesday, together with this additional problem.

Let $\chi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be any operator obeying the following properties:

- $\mathbb{R}$-linearity: $\chi(\alpha f + \beta g) = \alpha\chi(f) + \beta\chi(g)$ for $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(\mathbb{R})$.
- Product rule: $\chi(f \cdot g) = \chi(f) \cdot g + f \cdot \chi(g)$.
- Identity: $\chi(x) = 1$, where $x \in C^\infty(\mathbb{R})$ denotes the identity function. (i.e. $\chi(f) = 1$ when $f(x) = x$.)

Problem 8.

- Prove that $\chi(1) = 0$, where $1 \in C^\infty(\mathbb{R})$ denotes the constant function with value 1.
- Prove that for any $a \in \mathbb{R}$,
  $$\chi(f)|_{x=a} = f'(a).$$
  Conclude that $\chi = \frac{d}{dx}$. Hint: Use a first-order Taylor polynomial.
- Suppose that the "Identity" property no longer holds. Rather than $\chi(x) = 1$, suppose $\chi(x) = g$ for some $g \in C^\infty(\mathbb{R})$. Generalize the above result.