# Calculus on Manifolds
# Problem Set 2
# Due Friday, Jan. 20, 2012, beginning of class

## Functors

Refer to the definitions and examples on p. 27. Although we're skipping ahead here, this page is self-contained.

Take the category $\mathcal{C}$ to be

> The category of open sets in Euclidean spaces, where the morphisms are the smooth maps.

Take the category $\mathcal{V}$ to be

> The category of [real] vector spaces, where the morphisms are the linear maps.

We will verify some basic facts about the functor $C^\infty : \mathcal{C} \to \mathcal{V}$. This functor is defined on objects $U \in \mathcal{C}$ by

$$C^\infty(U) := C^\infty(U; \mathbb{R}).$$

We also need to specify how the $C^\infty$ functor is defined on morphisms. This functor is contravariant, so for $\phi : U_1 \to U_2$, the induced map is some linear map of vector spaces $\phi^* : C^\infty(U_2) \to C^\infty(U_1)$. Specifically, it is defined by

$$\phi^*(f) := f \circ \phi.$$

**Problem 1.**

- Verify that $\phi^*$ is indeed a linear map of real vector spaces.
- Verify that the identity map $\mathrm{Id}_{U_1} : U_1 \to U_1$ induces the identity map $C^\infty(U_1) \to C^\infty(U_1)$.
- Verify that the composition map $\psi \circ \phi$

  $$U_1 \xrightarrow{\phi} U_2 \xrightarrow{\psi} U_3, \qquad U_1 \xrightarrow{\psi \circ \phi} U_3$$

  induces the composition $\phi^* \circ \psi^*$

  $$C^\infty(U_1) \xleftarrow{\phi^*} C^\infty(U_2) \xleftarrow{\psi^*} C^\infty(U_3), \qquad C^\infty(U_1) \xleftarrow{\phi^* \circ \psi^*} C^\infty(U_3).$$

  In other words, prove that $(\psi \circ \phi)^* = \phi^* \circ \psi^*$. (This is the sense in which a functor is a homomorphism with respect to composition of functions.)

**Definition.** A smooth (smooth = $C^\infty$) map $\phi : U_1 \to U_2$ is said to be a *diffeomorphism* if there exists a smooth two-sided inverse map $\phi^{-1} : U_2 \to U_1$. Specifically,

$$\phi^{-1} \circ \phi = \mathrm{Id}_{U_1}, \text{ and}$$
$$\phi \circ \phi^{-1} = \mathrm{Id}_{U_2}.$$

**Problem 2.** Prove that a diffeomorphism induces an isomorphism of vector spaces.

(Note how your resulting proof is a formal consequence of the aforementioned properties. You don't need any additional facts about smooth functions or vector spaces. These sorts of vacuous category-theoretic proofs are affectionately called "abstract nonsense.")

## Quotient spaces

Suppose $V$ is a finite-dimensional real vector space, and $W \subset V$ is a vector subspace. Recall the definition of the quotient space $V/W$. (Wikipedia)

There is a natural "quotient map" $q : V \to V/W$ given by $v \mapsto [v]$.

**Problem 3.**

- Compute both the kernel and image of $q$. Use this to compute $\dim(V/W)$.
- Using $q$, construct an exact sequence which resolves the subspace $W$.

  Note: for deep technical reasons, such a resolution is not useful for defining cohomology.

Suppose further that $V$ comes equipped with some Euclidean inner product denoted by $\langle \bullet, \bullet \rangle$. The orthocomplement of $W$ is defined as

$$W^\perp := \{v \in V | \langle v, w \rangle = 0 \text{ for all } w \in W\}.$$

**Problem 4.** Prove that the restriction of $q$ to $W^\perp$ is an isomorphism.

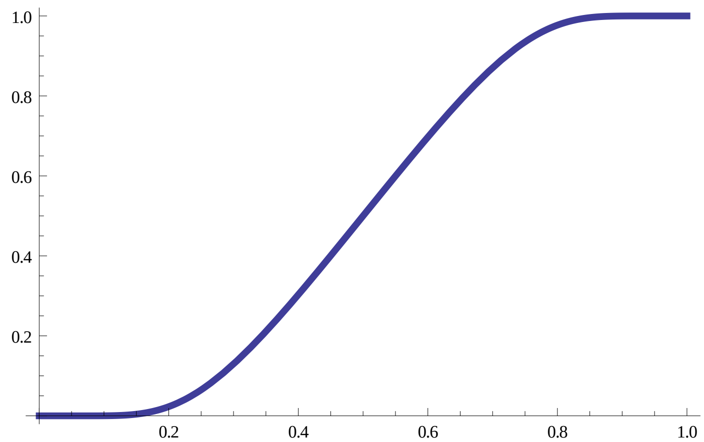
## Cutoff functions

It's useful to be able to craft functions which have carefully prescribed behavior. Specifically, we would like a "cutoff function" which interpolates between the constant functions zero and one. To an engineer, such a function looks like the voltage of a digital signal.

Specifically, we want a function $\psi(t)$ which satisfies

$$\begin{cases} \psi(t) = 0 & \text{if } t \le 0, \\ \psi(t) = 1 & \text{if } t \ge 1, \\ 0 < \psi(t) < 1 & \text{otherwise.} \end{cases} \tag{1}$$

Constructing such a function can be reduced to a simpler problem as follows:




**Problem 5.** Show that if $\omega(t)$ satisfies

$$\begin{cases} \omega(t) = 0 & \text{if } t \leq 0, \\ \omega(t) > 0 & \text{if } t > 0, \end{cases} \tag{2}$$

then the function

$$\psi(t) := \frac{\omega(t)}{\omega(t) + \omega(1-t)}$$

satisfies (1).

For example,

$$\omega(t) = t + |t| \implies \psi(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ t & \text{if } 0 \leq t \leq 1, \\ 1 & \text{if } t \geq 1. \end{cases}$$

This particular $\omega$ (and hence the corresponding $\psi$) is $C^0$, but not $C^1$. We would like to remain in the $C^\infty$ realm, so we are led to the

**Question** Is it possible to have a $C^\infty$ function satisfying (2) (and hence (1))?

Recall the following inclusions of function spaces:

$$\{\text{polynomials}\} \subset C^\infty(\mathbb{R}) \subset C^k(\mathbb{R}).$$

A more refined question is

**Question** How far left can we go in this diagram and still have a function satisfying (2)?

Recall that $C^k$ denotes the space of functions whose $k$-th derivatives are continuous, where $k$ is any nonnegative integer. It's easy to verify that

$$\omega_k(t) := (t + |t|)\, t^k \in C^k(\mathbb{R}) \tag{3}$$

satisfies (2). However, none of these functions are $C^\infty$.

**Problem 6.** Prove that it's impossible for a polynomial function to satisfy (2). Are rational functions any different?

Now let's analyze the function

$$\omega_\infty(t) := \begin{cases} \exp(-1/t) & \text{if } t > 0, \\ 0 & \text{if } t \le 0. \end{cases}$$

**Problem 7.** Read and understand the proof of Lemma A.2 in Appendix A. (Hint: the limit comes from rewriting the $n$-th derivative $\omega_\infty^{(n)}(0)$ as a difference quotient.) Do not rewrite this proof. Instead, compute for $\omega_\infty$ the Maclaurin polynomials and remainder functions $S_k$, as described in Problem Set 1.

## Decomposable tensors

**Problem 8** (graduate level). Prove that a tensor $t \in V \otimes W$ is decomposable iff the matrix of components has rank one. Hint: you should choose good basis vectors for both $V$ and $W$.

**Extra credit:** More generally, show that the minimal number of terms required to express a tensor coincides with the matrix rank. Hint: choose your bases for $V$ and $W$ based on the singular value decomposition:

http://en.wikipedia.org/wiki/Singular_value_decomposition#Geometric_meaning

## Tensor calculus

Let $W$ be a vector space, which itself could be a tensor product. For $\phi \in C^\infty(U, W)$, we extend the definition of total derivative as follows. Choose a basis $\{w_j\}$ of $W$, and write $\phi = \sum \phi_j w_j$. Then

$$D\phi := \sum_{i,j} \frac{\partial \phi_j}{\partial x_i} \varepsilon_i \otimes w_j.$$

**Problem 9.** Compute

$$D\left( (x_1^2 + x_2 x_3 \, \varepsilon_1 \otimes \varepsilon_2) \otimes (x_2 + x_1 \varepsilon_1) \right).$$

Warning: the product rule for $D$ doesn't work the way you might think, because the derivative always tensors in from the *left*, i.e. the definition of $D\phi$ involves $\varepsilon_i \otimes w_j$ rather than $w_j \otimes \varepsilon_i$.