The last problem set!

Consider $[\omega] \in H^p(M^n)$. Let K^p be any closed (compact w/o boundary) oriented manifold. Suppose we also have a fixed smooth map $k : K \to M$. We define the pairing

$$\langle k, [\omega] \rangle := \int_K k^*(\omega) \in \mathbb{R}.$$

Problem 1. Suppose $k_0, k_1 : K \to M$ are smoothly homotopic, through some smooth homotopy $\tilde{k} : [0,1] \times K \to M$.

Show that $\langle k_0, [\omega] \rangle = \langle k_1, [\omega] \rangle$.

Hint: $\partial([0,1] \times K) = K \sqcup \overline{K}$, where $\overline{K}$ denotes the manifold K equipped with the opposite orientation.

More generally, suppose K_0 and K_1 are closed oriented p-dimensional manifolds. We say that two maps $k_0 : K_0 \to M$ and $k_1 : K_1 \to M$ are *bordant* when there is some $(p+1)$-dimensional manifold $\tilde{K}$ (which is compact, oriented, and *with* boundary) and a map $\tilde{k} : \tilde{K} \to M$ such that $\partial \tilde{K} = K_0 \sqcup \overline{K}_1$ and $\tilde{k}|_{K_i} = k_i$ for $i = 1, 2$.

Problem 2. Show that $\langle k_0, [\omega] \rangle = \langle k_1, [\omega] \rangle$ whenever k_0 and k_1 are bordant. Conclude that $\langle [k_0], [\omega] \rangle$ is well-defined, where $[k_0] \sim [k_1]$ denotes the bordism class.

Note: The notion of *homology* $H_p(M)$ is very similar to that of bordism. Roughly speaking, one considers finite linear combinations of bordisms, where the K's are required to be simplices (hyper-triangles). The "universal coefficient theorem" states that $H^p(M) = H_p(M)^*$, so that cohomology is the dual vector space of homology.

Earn your wizard hat

Problem 3 (M&T #4.1)**.** Consider a commutative diagram of vector spaces and linear maps with exact rows

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{d} & A_2 & \xrightarrow{d} & A_3 & \xrightarrow{d} & A_4 & \xrightarrow{d} & A_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
B_1 & \xrightarrow{d} & B_2 & \xrightarrow{d} & B_3 & \xrightarrow{d} & B_4 & \xrightarrow{d} & B_5
\end{array}$$

Show that

$$\begin{cases} f_4 \text{ injective} \\ f_1 \text{ surjective} \\ f_2 \text{ injective} \end{cases} \implies f_3 \text{ injective.}$$

Show that

$$\begin{cases} f_2 \text{ surjective} \\ f_5 \text{ injective} \\ f_4 \text{ surjective} \end{cases} \implies f_3 \text{ surjective.}$$

Conclude that if f_1, f_2, f_4, f_5 are isomorphisms, then so is f_3. (This assertion is called the 5-lemma, and is a key step in the proof of the famous Poincaré duality theorem: M oriented $\implies H^p(M) \cong H_c^{n-p}(M)^*$.)

Note: Although you could easily find the answer online, I urge you to resist the temptation. Instead, let me help you get started.

For the first part, we wish to show that f_3 is injective, i.e. that if $f_3(x) = 0$, then $x = 0$.

We begin with the arrow of f_3 mapping $x \in A_3$ to zero:

$$\begin{array}{c} x \\ \big\downarrow{\scriptstyle f_3} \\ 0 \end{array}$$

Our goal is to show that $x = 0$. Now we start filling out the diagram:

$$\begin{array}{ccc} x & \longmapsto & dx \\ \big\downarrow{\scriptstyle f_3} & & \big\downarrow{\scriptstyle f_4} \\ 0 & \longmapsto & 0 \end{array}$$

By commutativity, we see that $f_4(dx) = 0$. By injectivity of f_4, we conclude that $dx = 0$. This says that x is in the kernel of d. The top row is exact, so the kernel of d is the image of the previous d. Thus $x = d\tilde{x}$ for some $\tilde{x} \in A_2$:

$$\begin{array}{ccccc} \tilde{x} & \overset{d}{\longmapsto} & x & \overset{d}{\longmapsto} & 0 \\ & & \big\downarrow{\scriptstyle f_3} & & \big\downarrow{\scriptstyle f_4} \\ & & 0 & \longmapsto & 0 \end{array}$$

We are halfway to injectivity; keep filling things out, and you will eventually conclude that $x = 0$. The second proof that f_3 is surjective is similar.

Problem 4. Let $0 \to A^0 \overset{d^0}{\to} A^1 \overset{d^1}{\to} \cdots \overset{d^{n-1}}{\to} A^n \to 0$ be a chain complex, and assume that $\dim A^i < \infty$. The *Euler characteristic* of $A^\bullet$ is defined by

$$\chi(A^\bullet) := \sum_{i=0}^{n} (-1)^i \dim A^i.$$

Show that

$$\chi(A^\bullet) = \sum_{i=0}^{n} (-1)^i \dim H^i.$$

In particular, conclude that $\chi(A^\bullet) = 0$ whenever $A^\bullet$ is exact.

Hint: Use the following short exact sequences, and count dimensions.

SES associated to the map d^i:

$$0 \to \ker d^i \to A^i \to \operatorname{im} d^i \to 0,$$

SES definition of cohomology:

$$0 \to \operatorname{im} d^{i-1} \to \ker d^i \to H^i \to 0.$$

(See my “glossary” handout for more details.)

Warning: This was originally based on M&T 4.4, but you should ignore their confusing hint. The method I suggest is more straightforward.

Note: This result is useful when applied to long exact sequences. However, if we take $A^\bullet$ to be the de Rham complex $A^\bullet = \Omega^\bullet(M)$, then $\dim A^i = \dim \Omega^i(M) = \infty$, in which case this result is meaningless. There are other models of cohomology (namely simplicial cohomology) where $\dim(A^i) = \#i$-simplices. As a special case, one obtains the classical formula $\chi = V - E + F$ for any triangulation of a surface.

Algebraic topology

Problem 5. Suppose $m, n > 0$ and $m \neq n$. Use a cohomology functor to prove that if $f : S^m \to S^n$ and $g : S^n \to S^m$, then $f \circ g : S^n \to S^n$ is not homotopic to Id_{S^n}.

Hint: For $n > 0$,

$$H^p(S^n) = \begin{cases} \mathbb{R} & \text{if } p = 0, n, \\ 0 & \text{else.} \end{cases}$$

Problem 6. Let $M^3 \subset \mathbb{R}^3$ be a solid torus. (Think of a donut, laying flat on a table.) Give an orientation form for M^3. What does this tell you about the orientability of $T^2 = \partial M^3$? What does this tell you about $H^2(T^2)$? Also, what is $H^0(T^2)$?

Optional exercises with Mayer-Vietoris

Cohomology of T^2

Let U be the open subset consisting of the lower 2/3 of T^2 (thought of as the surface of a donut laying flat on a table). Let V be the open subset consisting of the upper 2/3. Observe that each of U and V is an open cylinder, $U \cup V = T^2$, and $U \cap V$ is two open cylinders.

Recall that an open cylinder is diffeomorphic to $\mathbb{R}^2 - \{0\}$, and

$$H^p(\mathbb{R}^2 - \{0\}) = \begin{cases} \mathbb{R} & \text{if } p = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 7. Use the Mayer-Vietoris sequence of U and V to determine the cohomology of T^2. You will need your answer to Problem 6.

Problem 8. Show that if $f : S^2 \to T^2$ and $g : T^2 \to S^2$, then $f \circ g : T^2 \to T^2$ is not homotopic to Id_{T^2}.