Suppose $U \subset \mathbb{R}^n$ is an open set which is star-shaped with respect to $\vec{0}$, and $[\alpha] \in H^p(U)$ with $p > 0$. Recall that

$$H^p(U) = \frac{Z^p(U)}{B^p(U)} = \frac{\text{cocycles}}{\text{coboundaries}} = \frac{\text{closed forms}}{\text{exact forms}} = \frac{\ker d}{\text{image}\, d} = \frac{\{\alpha \in \Omega^p(U) \mid d\alpha = 0\}}{\{d\eta \in \Omega^p(U) \mid \eta \in \Omega^{p-1}(U)\}}.$$

Any $x \in H^p(U)$ is $x = [\alpha]$ where $\alpha \in \Omega^p(U)$ is a representative which satisfies $d\alpha = 0$, and $[\alpha_1] = [\alpha_2]$ iff $\alpha_1 - \alpha_2 = d\eta$ for some $\eta \in \Omega^{p-1}(U)$.

We want to show that $[\alpha] = 0$, or equivalently $\alpha = d\eta$ for some η, whenever $d\alpha = 0$. We write

$$\alpha = \sum_I \alpha_I(x)\, dx_I.$$

Consider $\phi(t,x) := tx$, so that

$$\phi : [0,1] \times U \to U.$$

(That U is star-shaped with respect to zero is equivalent to the image of ϕ being contained in U.)

Note that we can treat t as either a parameter or coordinate. As a parameter, we get an interval's worth of functions

$$\phi_t : U \to U \;\text{ for each } t \in [0,1].$$

Define $\omega := \phi^*(\alpha)$. Let's compute this one piece at a time. Note that $\phi^*(dx_i) = d(tx_i) = t\,dx_i + x_i\,dt$. Slightly more generally,

$$\begin{aligned}\phi^*(dx_i \wedge dx_j \wedge dx_i) &= (t\,dx_i + x_i\,dt) \wedge (t\,dx_j + x_j\,dt) \wedge (t\,dx_k + x_k\,dt)\\ &= \underbrace{t^3\,dx_i \wedge dx_j \wedge dx_k}_{\phi_t^*(dx_i \wedge dx_j \wedge dx_k)} + dt \wedge (x_i\,dx_j \wedge dx_k - x_j\,dx_i \wedge dx_k + x_k\,dx_i \wedge dx_j).\end{aligned}$$

Finally,

$$\omega = \phi^*(\alpha) = \sum_I \phi^*(\alpha_I(x))\,\phi^*(dx_I) = \sum_I \alpha_I(tx)\,\phi^*(dx_I),$$

which we write as

$$\omega = \underbrace{\sum f_I(t,x)\,dx_I}_{\phi_t^*(\alpha)} + dt \wedge \sum g_J(t,x)\,dx_J,$$

and we know that

$$f_I = t^p \alpha_I(tx).$$

Note that $d\omega = 0$ since $d\omega = d(\phi^*(\alpha)) = \phi^*(d\alpha) = \phi^*(0) = 0$.

$$0 = d\omega = \sum_I \left(\sum_i \frac{\partial f_I}{\partial x_i}(t,x)\,dx_i + \frac{\partial f_I}{\partial t}(t,x)\,dt \right) \wedge dx_I - dt \wedge \sum_J \left(\sum_i \frac{\partial g_J}{\partial x_i}(t,x)\,dx_i + \cancel{\frac{\partial g_J}{\partial t}(t,x)\,dt} \right) dx_J.$$

Looking at the coefficient of dt, we see that

$$0 = \sum_I \frac{\partial f_I}{\partial t}(t,x)\,dx_I - \sum_{J,i} \frac{\partial g_J}{\partial x_i}(t,x)\,dx_i \wedge dx_J.$$

By the fundamental theorem of calculus,

$$
\begin{aligned}
\sum_I f_I(1,x)\,dx_I - \sum_I f_I(0,x)\,dx_I &= \sum_{J,i} \left(\int_0^1 \frac{\partial g_J}{\partial x_i}(t,x)\,dt \right) dx_i \wedge dx_J \\
\phi_1^*(\alpha) - \phi_0^*(\alpha) &= \sum_i dx_i \wedge \frac{\partial}{\partial x_i} \sum_J \left(\int_0^1 g(t,x)\,dt \right) dx_J \\
&= d\left(\sum_J \left(\int_0^1 g(t,x)\,dt \right) dx_J \right).
\end{aligned}
$$

For the left hand side, note that $\phi_1^*(\alpha) = \alpha$, and for $p > 0$, $\phi_0^*(\alpha) = 0$.

Define $\eta := \sum_J \left(\int_0^1 g(t,x)\,dt \right) dx_J \in \Omega^{p-1}(U)$. Then

$$\alpha = d\eta.$$

To generalize, write

$$\omega = \beta + dt \wedge \gamma,$$

where $\omega \in \Omega^p([0,1] \times U)$. Note that we can treat t as either a parameter or coordinate. As a parameter, for any fixed t_0, we can view $\beta(t_0) \in \Omega^p(U)$ and $\gamma(t_0) \in \Omega^{p-1}(U)$. Note that

$$\beta(t) = \phi_t^*(\alpha).$$

Alternatively, treating t as a coordinate, $\beta \in \Omega^p([0,1] \times U)$ and $\gamma \in \Omega^p([0,1] \times U)$. Correspondingly, there are two ways to compute the exterior derivative d_U or $d_{[0,1]\times U}$, related by

$$d_{[0,1]\times U}\,\omega = dt \wedge \frac{\partial \omega}{\partial t} + d_U \omega.$$

Note that

$$
\begin{aligned}
d_U \beta &= d\left(\beta(t)\right), \\
d_U \gamma &= d(\gamma(t)).
\end{aligned}
$$

We compute

$$
\begin{aligned}
0 = d\omega = d_{[0,1]\times U}\,\omega &= dt \wedge \frac{\partial \beta}{\partial t} + d_U \beta - dt \wedge \left(\cancel{dt \wedge \frac{\partial \gamma}{\partial t}} + d_U \gamma \right) \\
&= d_U \beta + dt \wedge \left(\frac{\partial \beta}{\partial t} - d_U \gamma \right).
\end{aligned}
$$

Define the operator $\hat{S} : \Omega^{p-1}([0,1] \times U) \to \Omega^{p-1}(U)$ by

$$\hat{S}(\omega_1 + dt \wedge \omega_2) := \int_0^1 \omega_2\,dt.$$

Then

$$0 = \hat{S}(d\omega) = \int_0^1 \left(\frac{\partial \beta}{\partial t} - d(\gamma(t)) \right) dt = \beta(1) - \beta(0) - d \int_0^1 \gamma(t)\,dt,$$

and because $\beta(t) = \phi_t^*(\alpha)$, we have $\beta(1) - \beta(0) = \phi_1^*(\alpha) - \phi_0^*(\alpha) = \alpha - 0$. Thus

$$\alpha = d \int_0^1 \gamma(t)\, dt.$$

Thus if we set

$$\eta = \int_0^1 \gamma(t)\, dt = \hat{S}(\beta + dt \wedge \gamma) = \hat{S}(\omega) = \hat{S}(\phi^*(\alpha)),$$

we get

$$\alpha = d\eta.$$

The operator $S_\phi := \hat{S} \circ \phi^*$ is called the *homotopy operator* corresponding to the homotopy ϕ. If $d\alpha = 0$, then we have

$$\alpha = dS_\phi \alpha.$$

For more general α not necessarily satisfying $d\alpha = 0$, we compute

$$dS_\phi \alpha = d\hat{S}(\beta + dt \wedge \gamma) = d \int_0^1 \gamma(t)\, dt = \int_0^1 d_U \gamma(t)\, dt.$$

Swapping the order, we find

$$\begin{aligned} S_\phi d\alpha &= \hat{S}\left(d_U \beta + dt \wedge \left(\frac{\partial \beta}{\partial t} - d_U \gamma\right)\right) \\ &= \int_0^1 \left(\frac{\partial \beta}{\partial t} - d_U \gamma\right) dt \\ &= \beta(1) - \beta(0) - \int_0^1 d_U \gamma(t)\, dt \\ & \phi_1^*(\alpha) - \phi_0^*(\alpha) - dS_\phi \alpha. \end{aligned}$$

Therefore,

$$\boxed{(dS_\phi + S_\phi d)\alpha = \phi_1^*(\alpha) - \phi_0^*(\alpha).}$$

This is the abstract characterization of a homotopy operator, and a major result!

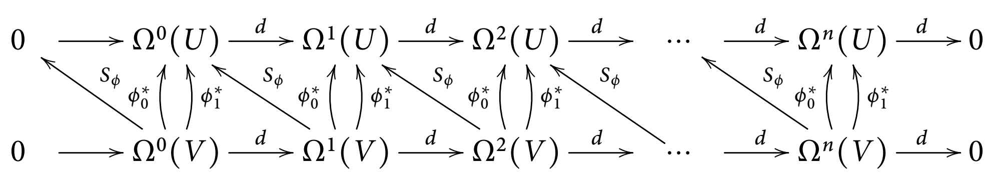
From here, we can conclude homotopy invariance of cohomology. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be any general open sets, and suppose $\phi : [0,1] \times U \to V$ is any smooth map, so that ϕ_0 is smoothly homotopic to ϕ_1. For any $[\alpha] \in H^p(U)$, we wish to show that $\phi_0^*([\alpha]) = \phi_1^*([\alpha])$. As above, define $S_\phi := \hat{S} \circ \phi^*$, so that $S_\phi : \Omega^p(V) \to \Omega^{p-1}(U)$. Then

$$\phi_1^*([\alpha]) - \phi_0^*([\alpha]) = [\phi_1^*(\alpha) - \phi_0^*(\alpha)] = \left[dS_\phi \alpha - S\cancel{d\alpha}\right] = \left[dS_\phi \alpha\right] = 0.$$

Thus

$$\phi_1^* = \phi_0^* \text{ on } H^p,$$

so homotopic maps are identical on cohomology.



Rather than doing explicit computations by hand, it often works quite well to instead study the abstract operators on chain complexes as above. This is the subject of homological algebra. We will study this after integration on manifolds.