Poincaré Lemma for $\mathrm{rot}(\vec{\phi})$ in two dimensions

Suppose $U \subset \mathbb{R}^2$ is open, and $\vec{\phi} \in C^\infty(U, \mathbb{R}^2)$. Last time we saw that

$$\vec{\phi} = \vec{\nabla} F \implies \mathrm{rot}(\vec{\phi}) = 0.$$

Thus $\mathrm{rot}(\vec{\phi}) = 0$ is a relation satisfied by all vector fields in the image of $\vec{\nabla} : C^\infty(U, \mathbb{R}) \to C^\infty(U, \mathbb{R}^2)$.

Furthermore, when U is star-shaped, then this is the only relation, i.e. the condition is sufficient:

$$\mathrm{rot}(\vec{\phi}) = 0 \implies \vec{\phi} = \vec{\nabla} F \text{ when } U \text{ is star-shaped.}$$

Because we observed that vector fields of the form $\vec{\nabla} F$ satisfy a relation, it's natural to ask a related question: Given a vector field $\vec{\phi} \in C^\infty(U, \mathbb{R}^2)$, does a function of the form $\mathrm{rot}(\vec{\phi}) \in C^\infty(U, \mathbb{R})$ satisfy any relations? In other words, given $g \in C^\infty(U, \mathbb{R})$, can we always find some $\vec{\phi} \in C^\infty(U, \mathbb{R}^2)$ such that $\mathrm{rot}(\vec{\phi}) = g$?

Whenever U is star-shaped, the answer is yes.

Lemma (Poincaré). *If $U \subset \mathbb{R}^2$ is a star-shaped open set, then for any $g \in C^\infty(U, \mathbb{R})$, one can find a vector field $\vec{\phi} \in C^\infty(U, \mathbb{R}^2)$ so that $g = \mathrm{rot}(\vec{\phi})$.*

Proof. Without loss of generality, by translation we may assume that U is star-shaped with respect to $\vec{0}$. Define a new function $\tilde{g} \in C^\infty(U, \mathbb{R})$ by

$$\tilde{g}(x_1, x_2) := \int_0^1 t\, g(tx_1, tx_2)\, dt.$$

I claim that if we choose

$$\vec{\phi}(x_1, x_2) := \begin{pmatrix} x_2\, \tilde{g}(x_1, x_2) \\ -x_1\, \tilde{g}(x_1, x_2) \end{pmatrix},$$

then $\mathrm{rot}(\vec{\phi}) = g$. The proof is another annoying computation, very similar to Theorem 1.4:

$$\begin{aligned}
\mathrm{rot}(\vec{\phi}) &= \frac{\partial}{\partial x_2}\left(x_2\, \tilde{g}(x_1, x_2)\right) - \frac{\partial}{\partial x_1}\left(-x_1\, \tilde{g}(x_1, x_2)\right) \\
&= 2\, \tilde{g}(x_1, x_2) + x_1 \frac{\partial}{\partial x_1} \tilde{g}(x_1, x_2) + x_2 \frac{\partial}{\partial x_2} \tilde{g}(x_1, x_2) \\
&= \int_0^1 \left(2t\, g(tx_1, tx_2) + x_1 t \frac{\partial}{\partial x_1} g(tx_1, tx_2) + x_2 t \frac{\partial}{\partial x_2} g(tx_1, tx_2)\right) dt \\
&= \int_0^1 \left(2t\, g(tx_1, tx_2) + x_1 t^2 \frac{\partial g}{\partial x_1}(tx_1, tx_2) + x_2 t^2 \frac{\partial g}{\partial x_2}(tx_1, tx_2)\right) dt \\
&= \int_0^1 \frac{d}{dt}\left(t^2\, g(tx_1, tx_2)\right) dt \\
&= t^2\, g(tx_1, tx_2)\Big|_0^1 \\
&= g(x_1, x_2).
\end{aligned}$$

□

Thus if $U \subset \mathbb{R}^2$ is star-shaped, then there are no relations on $\mathrm{rot}(\vec{\phi})$, so

$$C^\infty(U,\mathbb{R}) \xrightarrow{\vec{\nabla}} C^\infty(U,\mathbb{R}^2) \xrightarrow{\mathrm{rot}} C^\infty(U,\mathbb{R}) \longrightarrow 0$$

is an exact sequence which resolves the subspace of constant functions in $C^\infty(U,\mathbb{R})$.

Similarly, if $U \subset \mathbb{R}^3$ is star-shaped, then

$$C^\infty(U,\mathbb{R}) \xrightarrow{\vec{\nabla}} C^\infty(U,\mathbb{R}^3) \xrightarrow{\overrightarrow{\mathrm{rot}}} C^\infty(U,\mathbb{R}^3) \xrightarrow{\mathrm{div}} C^\infty(U,\mathbb{R}) \longrightarrow 0$$

is an exact sequence which resolves the subspace of constant functions in $C^\infty(U,\mathbb{R})$.

The goal of the next several lectures will be to develop the tensor calculus necessary to generalize these results to $\mathbb{R}^n$. In particular, we will create a resolution

$$\Omega^0(U) \xrightarrow{d^0} \Omega^1(U) \xrightarrow{d^1} \Omega^2(U) \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} \Omega^n(U) \longrightarrow 0$$

of constant functions, which generalizes the above cases $n = 2$ and $n = 3$.