Review of Stokes' Theorem with boundary

Suppose M^n is oriented, and $\eta \in \Omega_c^{n-1}(M)$. Partitions of unity work the same way on manifolds with boundary. Thus we can reduce

$$\int_{\partial M} i^*(\eta) = \int_M d\eta$$

to

$$\int_{\mathbb{R}^{n-1}} i^*(\eta) = \int_{\mathbb{R}^n_-} d\eta, \quad \eta \in \Omega_c^{n-1}(\mathbb{R}^n_-).$$

(Recall that $\partial\mathbb{R}^n_- = \mathbb{R}^{n-1}$.)

In coordinates, we have $\eta = \sum_{i=1}^n (-1)^{i-1}\, \eta_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$.

The support of η is a closed and bounded subset of $\mathbb{R}^n_-$, so it is contained in some large half-ball.

The map i^* sets $x_1 = 0$, so

$$i^*(\eta) = \eta_1(0, x_2, \ldots, x_n)\, dx_2 \wedge \cdots \wedge dx_n + 0.$$

Computing,

$$\int_{\mathbb{R}^n_-} d\eta = \sum_{i=1}^n \int_{x_n=-\infty}^{\infty} \cdots \int_{x_2=-\infty}^{\infty} \int_{x_1=-\infty}^{0} \cdots \frac{\partial \eta_i}{\partial x_i}\, dx_1 \cdots dx_n.$$

We can reorder the integrals by Fubini's Theorem. We want to integrate x_i first:

$$\int_a^b \frac{\partial \eta_i}{\partial x_i} dx_i = \eta_i|_{x_i=b} - \eta_i|_{x_i=a}.$$

By compact support, as $a \to -\infty$, the hyperplane $x_i = a$ moves outside the support, so $\eta_i|_{x_i=a} \equiv 0$. When $i \neq 1$, the same thing happens as $b \to +\infty$. The only remaining contribution is from the term

$$\begin{aligned}
\int_{\mathbb{R}^n_-} d\eta &= \int_{x_n=-\infty}^{\infty} \cdots \int_{x_2=-\infty}^{\infty} \left[\int_{x_1=-\infty}^{0} \frac{\partial \eta_1}{\partial x_1} dx_1 \right] dx_2 \cdots dx_n \\
&= \int_{\mathbb{R}^{n-1}} (\eta_1|_{x_1=0} - 0)\, dx_2 \cdots dx_n \\
&= \int_{\mathbb{R}^{n-1}} i^* \eta.
\end{aligned}$$

Example 1. Fundamental Theorem of Calculus:

$$M = [a, b], \quad \partial M = \{a\} \sqcup \{b\}.$$

M is oriented by $dt \in \Omega^1([a, b])$. An orientation on ∂M is a nowhere-vanishing element of $\Omega^0(\{a\} \sqcup \{b\})$, so it determies a $\pm$ sign for each point. The outward conormal for b is dt, while the outward conormal for a is $-dt$. Using the outward-first convention:

$$\begin{aligned}
dt &= dt \cdot (+1) \implies \text{sign}(b) = +1, \\
&= (-dt) \cdot (-1) \implies \text{sign}(a) = -1.
\end{aligned}$$

Example 2. Cylinders for Problem #8:

If M is a manifold without boundary ($\partial M = \varnothing$), and ω_0 is an orientation form on M, the *cylinder of* M is $[0,1] \times M$ with orientation form $dt \wedge \omega_0$. Then

$$\partial\left([0,1] \times M\right) = \left(\partial [0,1]\right) \times M = \left(\{0\} \sqcup \{1\}\right) \times M = M_0 \sqcup M_1.$$

The outward conormal for M_1 is dt, while the outward conormal for M_0 is $-dt$. Thus

$$\begin{aligned} dt \wedge \omega_0 &\implies (M_1, [\omega_0]) \\ =(-dt) \wedge (-\omega_0) &\implies (M_0, [-\omega_0]) \,. \end{aligned}$$

Thus M_1 is a copy of M, while M_0 is M with the orientation reversed. We write this as $M_0 = \overline{M}$ and $M_1 = M$. Then Stokes' theorem tells us that if $\eta \in \Omega^{n-1}([0,1] \times M)$, then

$$\int_{[0,1]\times M} d\eta = \int_{M_0 \sqcup M_1} \eta = \int_{\overline{M}} i_0^*(\eta) + \int_M i_1^*(\eta) = \int_M i_1^*(\eta) - \int_M i_0^*(\eta).$$

Example 3. Here we use Stokes' Theorem without boundaries. Suppose M^n is closed (=compact, without boundary) and oriented. Then $H^n(M) \neq 0$.

Proof. Since M is compact, $\Omega^p(M) = \Omega_c^p(M)$. Since M is oriented, there exists an orientation form $\omega \in \Omega_c^n(M)$ which is a positive multiple of $dx_1 \wedge \cdots \wedge dx_n$ in each coordinate chart. Thus $\int_M \omega > 0$.

Note that ω is closed automatically since $d\omega \in \Omega^{n+1}(M) = 0$. Thus it determines a cohomology class $[\omega] \in H^n(M)$. (Warning: previously I have been using the notation $[\omega] \sim [f\omega]$, $f > 0$ to denote an orientation. Now I mean cohomology: $[\omega] \sim [\omega + d\eta]$.)

For contradiction, suppose $[\omega] = 0 \in H^n(M)$. This is equivalent to $\omega = d\eta$ for some $\eta \in \Omega^{n-1}(M)$. Then

$$0 < \int_M \omega = \int_M d\eta = 0 \quad (\text{ since } \partial M = \varnothing).$$

□

Theorem 4. *Suppose M is compact and oriented. There does not exist a smooth map $f : M \to \partial M$ which fixes the boundary, i.e. $f|_{\partial M} = \mathrm{Id}_{\partial M}$. For example, there is no map from the ball to its boundary sphere which fixes the boundary sphere.*

Proof. M oriented $\implies \partial M$ oriented. $\partial M \subset M$ closed $\implies \partial M$ compact. There exists an orientation form $\omega_0 \in \Omega^{n-1}(\partial M) = \Omega_c^{n-1}(\partial M)$. Using a positive atlas for ω_0, the coordinate expressions are positive multiples of $dx_2 \wedge \cdots \wedge dx_n$. Thus

$$\int_{\partial M} \omega_0 > 0.$$

Now we argue by contradiction.

$$0 < \int_{\partial M} \omega_0 = \int_{\partial M} \mathrm{Id}_{\partial M}^*(\omega_0) = \int_{\partial M} f^*(\omega_0) = \int_M d\, f^*(\omega_0) = \int_M f^*(d\omega_0) = \int_M f^*(0) = 0,$$

where we used $d\omega_0 \in \Omega^n(\partial M) = 0$ since ∂M only has dimension $n-1$. □

How do we make sense of what just happened? I don't know exactly, but here is a paradigm shift which might help.

Doing topology with category theory

We want to think of things categorically. This means by using commutative diagrams and functors.

Definition. A *commutative diagram* is a diagram where any composition is independent of the path taken.

Example 5. The fundamental property of exterior derivatives $d\,f^*(\omega) = f^*(d\omega)$ is encoded by the diagram

$$f : M \to N \implies \begin{array}{ccc} \Omega^p(N) & \xrightarrow{f^*} & \Omega^p(M) \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\ \Omega^{p+1}(N) & \xrightarrow{f^*} & \Omega^{p+1}(M) \end{array}$$

Example 6. Stokes' Theorem:

$$\begin{array}{ccc} \Omega_c^{n-1}(M) & \xrightarrow{i^*} & \Omega_c^{n-1}(\partial M) \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \int_{\partial M}} \\ \Omega_c^{n}(M) & \xrightarrow{\int_M} & \mathbb{R} \end{array}$$

Let's reprove our theorem categorically.

Proof. The hypothesis $f|_{\partial M} = \mathrm{Id}_{\partial M}$ can be rephrased as the commutative diagram

$$f|_{\partial M} = \mathrm{Id}_{\partial M} \implies \partial M \overset{i}{\hookrightarrow} M \xrightarrow{f} \partial M \, , \quad (f \circ i = \mathrm{Id}_{\partial M}).$$

Since M and ∂M are compact, $\Omega_c^p = \Omega^p$. By example of the orientation form on ∂M, we know that the following map is nonzero:

$$\begin{array}{c} \Omega^{n-1}(\partial M) \\ \downarrow{\scriptstyle \int_{\partial M} \neq 0} \\ \mathbb{R} \end{array}$$

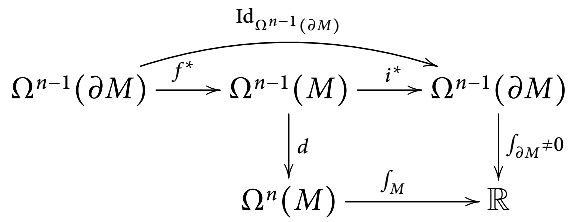
The strategy is to assemble everything we know like LEGOs. Stokes':

$$\begin{array}{ccc} \Omega^{n-1}(M) & \xrightarrow{i^*} & \Omega^{n-1}(\partial M) \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \int_{\partial M} \neq 0} \\ \Omega^{n}(M) & \xrightarrow{\int_M} & \mathbb{R} \end{array}$$

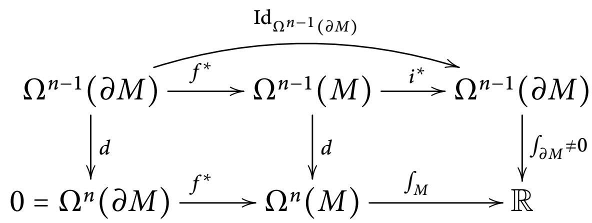
The upper row suggests applying the functor Ω^{n-1} :

$$\partial M \overset{i}{\hookrightarrow} M \xrightarrow{f} \partial M \;\; (\mathrm{Id}_{\partial M}) \overset{\Omega^{n-1}}{\implies} \Omega^{n-1}(\partial M) \xrightarrow{f^*} \Omega^{n-1}(M) \xrightarrow{i^*} \Omega^{n-1}(\partial M) \;\; (\mathrm{Id}_{\Omega^{n-1}(\partial M)}) \, .$$

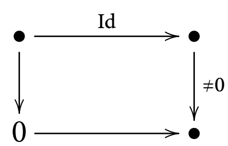
Putting it together:



Building further:



Now we have generated our contradiction:



□

These types of arguments are the essence of algebraic topology.