Let X be a closed oriented Riemannian manifold of dimension n, $E_i \to X$ Euclidean vector bundles for $i \in \{1,2\}$, and $D : \Gamma(E_1) \to \Gamma(E_2)$ an elliptic (linear) differential operator of order d.

Elliptic regularity

Exercise 1. Give a brief but careful proof that if β is any distributional section of E, and if $D\beta \in \Gamma(E_2)$, i.e. if $D\beta$ is a smooth section, then $\beta \in \Gamma(E_1)$, i.e. then β is smooth. Observe that since the zero section $0 \in \Gamma(E_2)$ is smooth, the subspace $\ker D$ does not depend on the choice of Sobolev completion

$$D : H^{s+d}(E_1) \to H^s(E_2),$$

i.e. although we enlarge the $\Gamma(E_i)$ to Sobolev completions, the subspace $\ker D$ does not shrink or grow.

Hint Use the elliptic estimate

$$\|\beta\|_{H^{s+d}} \leq C_{D,s}\left(\|D\beta\|_{H^s} + \|\beta\|_{H^s}\right)$$

in an inductive argument to show that the Sobolev degree s of β can be made arbitrarily large. (This argument is called "elliptic bootstrapping." When D is nonlinear, the procedure is similar, but when s is small, the estimate is more delicate due to correction terms.)

Poincaré inequality

Exercise 2. For any $s \in \mathbb{R}$, show that there is a constant $C'_{D,s}$ such that there is an elliptic estimate of the form

$$\|\beta\|_{H^{s+d}} \leq C'_{D,s}\,\|D\beta\|_{H^s},$$

for all $\beta \in H^{s+d}(X; E_1)$ which are L^2-orthogonal to $\ker D$.

Hint Fill in the details of the following argument.
First of all, the L^2 pairing between H^{s+d} and $\ker D$ is well-defined and finite, so the condition that $\beta \perp_{L^2} \ker D$ makes sense. Now we argue by contradiction. Suppose that no such constant $C'_{D,s}$ exists. Then there is a sequence $\{\beta_i\}$ with $\beta_i \perp_{L^2} \ker D$ such that the ratio $\|\beta_i\|_{H^{s+d}} / \|D\beta_i\|_{H^s} \to \infty$. Since this ratio is homogeneous, we can renormalize so that $\|\beta_i\|_{H^{s+d}} = 1$. After passing to a subsequence, we may assume that $\beta_i \overset{H^s}{\to} \beta$ for some $\beta \in H^s(X; E_1)$. Furthermore, $\beta \perp_{L^2} \ker D$. Finally, we compute $\|D\beta\|_{H^{s-d}} = 0$. This leads to an obvious contradiction.

Hint' Recall that the L^2 pairing extends to a perfect pairing between H^s and H^{-s}, so that $\langle \alpha, \beta \rangle_{L^2} \leq \|\alpha\|_{H^s}\,\|\beta\|_{H^{-s}}$. The kernel of any continuous linear map is a *closed* subspace (in particular, it is sequentially closed). Continuous maps of Banach spaces map take convergent sequences to convergent sequences, and limits to limits. The embeddings $H^{s+d} \hookrightarrow H^s$ and $H^s \hookrightarrow H^{s-d}$ are continuous and compact. To show that two distributions are equal, it suffices to show that their difference is zero under *any* norm.

Note In contrast with the proof from lecture 15, this proof is more tidy thanks to the language of distributions. Instead of interpreting $D\beta = 0$ in the "weak sense" where we must use smooth test functions α, we interpret $D\beta = 0$ directly in the distributional sense.

Coarse Laplacian

Exercise 3. .

- Let V and W be Euclidean vector spaces. For $v \in V$, show that the adjoint of the map

$$T_v : W \to V \otimes W$$
$$T_v(w) := v \otimes w$$

  is determined by

$$T_v^* : V \otimes W \to W$$
$$T_v^*(v' \otimes w') = \langle v, v' \rangle \, w'.$$

- If $E \to X$ is a Euclidean vector bundle equipped with a connection A, then $\nabla_A : \Gamma(E) \to \Gamma(T^*X \otimes E)$. Compute the principal symbol $\sigma(\nabla_A) : T^*X \to \operatorname{Hom}(E, T^*X \otimes E)$. Deduce the principal symbols of $\nabla_A^* : \Gamma(T^*X \otimes E) \to \Gamma(E)$ and the "coarse Laplacian" $\nabla_A^* \nabla_A : \Gamma(E) \to \Gamma(E)$.

  Remark. Recall from lecture 17 that the principal symbol of the Hodge Laplacian $\Delta : \Omega^k(X) \to \Omega^k(X)$ given by $\Delta = dd^* + d^*d$ is $\sigma(\Delta, x, p) = -|p|^2 \operatorname{Id}_{\Lambda^k}$. Taking the coarse Laplacian with $E = \Lambda^k T^*X$ and $A = \mathrm{LC}$ the Levi-Civita connection, we obtain a different second-order differential operator

$$\nabla_{\mathrm{LC}}^* \nabla_{\mathrm{LC}} : \Omega^k(X) \to \Omega^k(X).$$

- Based on the principal symbols, what can we conclude about the order of the differential operator given by the difference

$$\nabla_{\mathrm{LC}}^* \nabla_{\mathrm{LC}} - \Delta : \Omega^k(X) \to \Omega^k(X)?$$

Hint Recall that in a local trivialization,

$$\nabla_A(s^j e_j) = (\nabla + A)s^j e_j = (\partial_i s^j + A^j_{ik} s^k) \, dx^i \otimes e_j.$$

To compute the principal symbol $\sigma(\nabla_A, p, x)$, ignore all but the terms with the highest order derivatives, and replace ∂_i with the coordinate function p_i on the cotangent bundle. The principal symbol of a formal adjoint is given by $\sigma(D^*) = (-1)^d \sigma(D)^*$ where d is the degree of D. The symbol of a composition is the composition of symbols.

Note On $\Omega^0(X)$, we have $\Delta =$ ~~dd^*~~ $+ d^*d$, and we can identify ∇_{LC} with $d : \Omega^0(X) \to \Omega^1(X)$ and ∇_{LC}^* with $d^* : \Omega^1(X) \to \Omega^0(X)$. Thus $\nabla_{\mathrm{LC}}^* \nabla_{\mathrm{LC}} - \Delta = 0$. However, for $k > 0$ this difference is an operator involving the Riemannian curvature of X, given by the "Weitzenböck formula." This formula is a central lemma for compactness of the Seiberg-Witten moduli space.

Exercise 4. Suppose that $1 < p < \infty$ and $k \in \mathbb{Z}$ with $k \geq 1$ are such that $(k+1)/n - 1/p \geq 0$. (In particular, if $n = 4$ and $k = 1$, then $p \geq 2$.) Show that if $g \in \mathcal{G}^p_{k+1}$ and $A \in \mathcal{A}^p_k$, then

- $g \cdot A \in \mathcal{A}^p_k$,
- $F_A \in L^p_{k-1}(X; \Lambda^2 T^* X)$.

Hint Recall the local expressions

$$(g \cdot A)_\alpha = g_\alpha A_\alpha g_\alpha^{-1} - (dg_\alpha) g_\alpha^{-1},$$
$$(F_A)_\alpha = dA_\alpha + \tfrac{1}{2}\left[A_\alpha \wedge A_\alpha\right].$$

For the gauge transformation, note that L^p_k is a module for $\mathcal{G}^p_{k+1}$. (The borderline case requires the additional hypothesis that $g \in L^\infty$, which follows from the compactness of G.) For the curvature, show that if L^p_k is below the borderline, then the multiplication is continuous. If L^p_k is above the borderline, then the multiplication is obviously continuous. If L^p_k is borderline, then we can escape below the borderline via the embedding $L^p_k \hookrightarrow L^{p'}_k$ for any p' satisfying

$$1/p < 1/p' \leq \min\left(1, 1/p + \left((k+1)/n - 1/p\right)/2\right).$$

Then we obtain continuous multiplication

$$L^p_k \times L^p_k \hookrightarrow L^{p'}_k \times L^{p'}_k \to L^p_{k-1}.$$

Chern-Simons form

Exercise 5. Let X be a compact oriented 4-manifold with boundary. (There need not be any metric on X.) Let $P \to X$ be a principal bundle, and $A \in \mathcal{A}_P$.

- Verify that $d_A F_A = 0$.
- For $\alpha \in \Omega^p(X; \mathfrak{g}_{\mathrm{Ad}})$, verify that $d_A d_A \alpha = \left[F_A \wedge \alpha\right]$.
- Verify that $\Omega^\bullet(X; \mathfrak{g}_{\mathrm{Ad}})$ is a Lie superalgebra, i.e. if $\alpha, \beta, \gamma \in \Omega^\bullet(X; \mathfrak{g}_{\mathrm{Ad}})$, then

$$\left[\alpha \wedge \beta\right] = -(-1)^{|\alpha||\beta|}\left[\beta \wedge \alpha\right],$$
$$\left[\alpha \wedge \left[\beta \wedge \gamma\right]\right] = \left[\left[\alpha \wedge \beta\right] \wedge \gamma\right] + (-1)^{|\alpha||\beta|}\left[\beta \wedge \left[\alpha \wedge \gamma\right]\right],$$

  and in particular, if $a \in \Omega^1(X; \mathfrak{g}_{\mathrm{Ad}})$, then $\left[a \wedge \left[a \wedge a\right]\right] = 0$.

- Suppose also $A_0 \in \mathcal{A}_P$. Define $a := A - A_0 \in \Omega^1(X; \mathfrak{g}_{\mathrm{Ad}})$. Verify that

$$F_A = F_{A_0} + d_{A_0} a + \tfrac{1}{2}\left[a \wedge a\right], \text{ and}$$
$$d_{A_0} F_A = \left[F_A \wedge a\right].$$

- Let $\langle , \rangle_{\mathfrak{g}}$ be an invariant metric on $\mathfrak{g}$ so that $\langle [\chi_1, \chi_2], \chi_3 \rangle_{\mathfrak{g}} = -\langle \chi_2, [\chi_1, \chi_3] \rangle_{\mathfrak{g}}$. Verify that

$$\langle [\alpha \wedge \beta] \wedge \gamma \rangle_{\mathfrak{g}} = -(-1)^{|\alpha||\beta|} \langle \beta \wedge [\alpha \wedge \gamma] \rangle_{\mathfrak{g}} .$$

- For any $\omega, \eta \in \Omega^\bullet(X; \mathfrak{g}_{\mathrm{Ad}})$, verify that

$$d \langle \omega \wedge \eta \rangle_{\mathfrak{g}} = \langle d_A(\omega \wedge \eta) \rangle_{\mathfrak{g}}$$

for any choice of connection A by comparing the expressions in a local frame:

$$\begin{aligned} \left(d \langle \omega \wedge \eta \rangle_{\mathfrak{g}} \right)_\alpha &= d \langle \omega_\alpha \wedge \eta_\alpha \rangle_{\mathfrak{g}} = \langle d(\omega_\alpha \wedge \eta_\alpha) \rangle_{\mathfrak{g}} , \\ \left(\langle d_A (\omega \wedge \eta) \rangle_{\mathfrak{g}} \right)_\alpha &= \left\langle (d\omega_\alpha + [A_\alpha \wedge \omega_\alpha]) \wedge \eta_\alpha + (-1)^{|\omega|} \omega_\alpha \wedge (d\eta_\alpha + [A_\alpha \wedge \eta_\alpha]) \right\rangle_{\mathfrak{g}} . \end{aligned}$$

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- Reformulating some of the above identities in more useful form, we have

$$\begin{aligned} d_{A_0} a &= F_A - F_{A_0} - \tfrac{1}{2} [a \wedge a] , \\ d_{A_0} F_{A_0} &= 0, \\ d_{A_0} F_A &= [F_A \wedge a] , \\ d \langle \omega \wedge \eta \rangle_{\mathfrak{g}} &= \langle d_A(\omega \wedge \eta) \rangle_{\mathfrak{g}} . \end{aligned}$$

Verify that

$$\int_{\partial X} \langle a \wedge (F_{A_0} + F_A - \tfrac{1}{6} [a \wedge a]) \rangle_{\mathfrak{g}} = \int_X \langle F_A \wedge F_A - F_{A_0} \wedge F_{A_0} \rangle_{\mathfrak{g}} .$$

When $\partial X = \varnothing$, conclude that $\int_X \langle F_A \wedge F_A \rangle$ is independent of A.

Hint Recall that for $\omega \in \Omega^\bullet(X; \mathfrak{g}_{\mathrm{Ad}})$,

$$(d_A \omega)_\alpha = d\omega_\alpha + [A \wedge \omega_\alpha] .$$

Furthermore, if $\{\chi^i\}_{i=1}^{\dim \mathfrak{g}}$ is a local basis of sections of $\mathfrak{g}_{\mathrm{Ad}}$, then $\omega = \sum \chi^i \otimes \omega_i$. Thus

$$\begin{aligned} \omega \wedge \eta &= \sum \chi^i \otimes \chi^j \otimes \omega_i \wedge \eta_j \in \Omega^{|\omega|+|\eta|}(X; \mathfrak{g}_{\mathrm{Ad}} \otimes \mathfrak{g}_{\mathrm{Ad}}), \\ [\omega \wedge \eta] &= \sum \left[\chi^i, \chi^j \right] \otimes \omega_i \wedge \eta_j \in \Omega^{|\omega|+|\eta|}(X; \mathfrak{g}_{\mathrm{Ad}}), \\ \langle \omega \wedge \eta \rangle_{\mathfrak{g}} &= \sum \left\langle \chi^i, \chi^j \right\rangle \omega_i \wedge \eta_j \in \Omega^{|\omega|+|\eta|}(X). \end{aligned}$$