Problem 1. A Euclidean metric on $\mathbb{R}^k$ is represented by a positive-definite $k \times k$ matrix M, so that

$$\langle v, v \rangle := v^T M v \quad \text{for } v \in \mathbb{R}^k.$$

If we transform $\mathbb{R}^k$ by $v \mapsto gv$ for $g \in \mathrm{GL}(k)$, how must M transform so that $\langle v, v \rangle$ remains $\mathrm{GL}(k)$-invariant?

Now consider the standard metric $M = \mathrm{Id}_{k \times k}$. Show that the action of $\mathrm{GL}(k)$ on $\mathrm{Id}_{k \times k}$ is well-defined on the cosets $\mathrm{Met}(k) := \mathrm{GL}(k)/\mathrm{O}(k)$. Thus the action of $\mathrm{GL}(k)$ on $\mathrm{Id}_{k \times k}$ induces a map from $\mathrm{Met}(k)$ to positive-definite $k \times k$ matrices. Show that this map is a bijection.

Hint Construct a two-sided inverse from positive-definite matrices to $\mathrm{Met}(k)$.

Remark. If P is a principal $\mathrm{GL}(k)$ bundle, then by Čech theory, a reduction to $\mathrm{O}(k)$ corresponds to a section of $P \times_{\rho_L} \mathrm{Met}(k)$. If $E := P \times_{\rho_{\mathrm{st}}} \mathbb{R}^k$ is the associated vector bundle, then via the bijection above, a section of $P \times_{\rho_L} \mathrm{Met}(k)$ corresponds to a section of the bundle of positive-definite bilinear forms on the fibers of E, i.e. a Euclidean structure on E.

Problem 2. After proving the Hodge decomposition

$$\Omega^p(X) = (\text{image } \Delta) \oplus (\ker \Delta) = (\text{image } d) \oplus (\text{image } d^*) \oplus \mathcal{H}^p(X),$$

we defined the Green's operator

$$\begin{aligned} &G : \Omega^p(X) \to (\mathcal{H}^p(X)^\perp \subset \Omega^p(X)) \\ &\qquad G(\alpha) := \omega, \text{ where } \omega \in \mathcal{H}^p(X)^\perp \text{is the unique solution to } \Delta\omega = \alpha - \pi_{\mathcal{H}^p}(\alpha). \end{aligned}$$

First verify that in the Hodge decomposition, image d, image d^*, and $\mathcal{H}^p(X)$ are mutually orthogonal.

Next verify the claimed properties of G by showing that it

- is well-defined (i.e. $\omega \in \mathcal{H}^p(X)^\perp$ is *uniquely* specified by the defining equation).
- is linear.
- $\ker G = \mathcal{H}^p(X)$.
- is surjective, i.e. image $G = \mathcal{H}^p(X)^\perp$.
- commutes with d, d^*, and Δ. (Hint: first show that d and d^* commute with Δ).
- satisfies

$$\Delta G = G \Delta = \mathrm{Id}_{\Omega^p} - \pi_{\mathcal{H}^p}.$$

- is self-adjoint (Hint: show that $\langle\!\langle G(\alpha), \beta \rangle\!\rangle = \langle\!\langle \alpha, G(\beta) \rangle\!\rangle$ by using the Hodge decomposition $\beta = \Delta\eta + h$ where h is harmonic, and η is modified to be in $\mathcal{H}^p(X)^\perp$).

- is L^2-bounded, i.e. $\|G(\alpha)\|_{L^2} \le C\|\alpha\|_{L^2}$ for some constant C (Hint: recall the Poincaré inequality from lecture 15) Consequently, G extends to the L^2 completion $L^2(X; \Lambda^p(T^*X))$ of $\Omega^p(X)$.
- this completion of G is compact operator.

Problem 3. Let X be a closed oriented Riemannian n-manifold. For any fixed $\omega \in \Omega^p(X)$, consider the problem of finding an element $\omega + d\eta$ which minimizes the action

$$S(\omega + d\eta) := \tfrac{1}{2}\|\omega + d\eta\|_{L^2}^2 = \tfrac{1}{2}\int_X |\omega + d\eta|^2 \underbrace{d\text{vol}}_{\star 1} = \tfrac{1}{2}\int_X (\omega + d\eta) \wedge \star(\omega + d\eta).$$

If $\omega_0 = \omega + d\eta$ is such an element, then the variational derivative must vanish:

$$\left.\frac{d}{dt}\right|_{t=0} S(\omega_0 + d(t\eta)) = 0 \quad \forall \eta \in \Omega^{p-1}(X).$$

Using the vanishing of this variational derivative, determine the corresponding Euler-Lagrange equation for ω_0. Generalize to the case where X is compact, oriented with boundary.

Hint To deal with the case where X has a boundary, you will use the formula

$$\int_{\partial X} \alpha \wedge \star\beta = \langle\!\langle d\alpha, \beta \rangle\!\rangle_{L^2} - \langle\!\langle \alpha, d^*\beta \rangle\!\rangle_{L^2}$$

from Lecture 14. First consider η supported on the interior of X. This will determine the equation satisfied by ω_0 on the interior of X. With this equation in hand, now consider general $\eta \in \Omega^{p-1}(X)$ to obtain the boundary condition.

Problem 4. Let X be a connected closed oriented 4-manifold. Then $\chi(X) := \sum_i (-1)^i b_i(X)$, and $\sigma(X) = b^+(X) - b^-(X)$. Show that $\chi(X) + \sigma(X)$ is even. (The solution is very straightforward.)

Problem 5. Let X be a connected closed oriented 4-manifold such that $H_1(X;\mathbb{Z})$ has no 2-torsion (i.e. if $x \in H_1(X;\mathbb{Z})$ satisfies $2x = 0$, then $x = 0$). Show that every $w \in H^2(X;\mathbb{Z}_2)$ is the $\pmod 2$ reduction of some $\tilde{w} \in H^2(X;\mathbb{Z})$.

Hint Use the short exact sequence $0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{\pmod 2} \mathbb{Z}_2 \to 0$ and the associated long exact sequence

$$\cdots \to H^2(X;\mathbb{Z}) \xrightarrow{\pmod 2} H^2(X;\mathbb{Z}_2) \to H^3(X;\mathbb{Z}) \to \cdots.$$

Problem 6. Describe all isomorphism classes of $\mathrm{SO}(3)$ principal bundles $[P]$ over $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ in terms of the instanton number $k([P])$ and Stiefel-Whitney class $w_2([P])$.

Next, the Atiyah-Singer index theorem gives the expected dimension for the moduli space of anti-self-dual instantons associated to an isomorphism class $[P]$ of principal G-bundles:

$$\dim \mathcal{M}_{\text{ASD}}([P]) = 4\check{h}(\mathfrak{g})k([P]) - \tfrac{1}{2}(\chi(X) + \sigma(X))\dim G.$$

Given that $\check{h}(\mathfrak{so}(3)) = 2$, compute the expected dimension for each $[P]$.

Hint For any given $w_2([P]) \in H^2(X; \pi_1(\mathrm{SO}(3))) \simeq H^2(X; \mathbb{Z}_2)$, to compute $k([P]) \pmod 1$, lift $w_2([P])$ to some $\tilde{w}_2 \in H^2(X; \mathbb{Z})$ and compute

$$k([P]) \equiv -\tfrac{1}{4} \langle \tilde{w}_2 \smile \tilde{w}_2, [X] \rangle \pmod 1.$$

Problem 7. Give an example of a smooth simply-connected 4-manifold X, together with an isomorphism class $[P]$ of a smooth principal $\mathrm{SO}(3)$ bundle over X such that the topological charge of $[P]$ is an integer, but the structure group does not extend to $\mathrm{SU}(2)$.