Problem 1. Show that any bilinear form over $\mathbb{Z}$ which is unimodular and antisymmetric is equivalent to a direct sum of copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Hint Given the corresponding matrix Q in an arbitrary integer basis, outline an algorithm to transform Q into standard form. Beware that working over $\mathbb{Z}$ is more subtle than working over $\mathbb{R}$ or $\mathbb{C}$. Recall the matrix Q of a bilinear form transforms under $G \in \mathrm{GL}(n;\mathbb{Z})$ via

$$Q \mapsto G^T Q G.$$

Understand the effect when G is either a transposition matrix, or of the form

$$G = \begin{pmatrix} 1 & & & & & \\ & 1 & & & n & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

with a single nonzero off-diagonal entry. To understand how to use these operations, consider the greatest common divisor of the first row of Q. Conclude that we can bring the first row into standard form. From there, it is straightforward to fix the second row and split off a copy of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Problem 2. Consider the topological manifold $\mathrm{K3} \# 44\, S^2 \times S^2$. (Here, $44\, S^2 \times S^2$ is shorthand for $(S^2 \times S^2)^{\# 44}$.) List the three ways (including this one, and distinct up to permutation) that this topological manifold can be expressed as a connected sum of the standard simply-connected smooth 4-manifolds: K3, $\overline{\mathrm{K3}}$, $S^2 \times S^2$, $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$. Similarly, list the twelve ways of expressing $6\,\mathbb{CP}^2 \# 39\,\overline{\mathbb{CP}^2}$.

Hint To summarize some relevant facts, recall that by Freedman's Theorem, two closed, smooth, simply-connected 4-manifolds are homeomorphic iff their intersection forms are equivalent. Intersection forms are classified by rank, signature, and type (even/odd), or equivalently b^+, b^- and type. Finally, $Q_{\mathrm{K3}} = -2E_8 \oplus 3H$, $Q_{S^2\times S^2} = H$, $Q_{\mathbb{CP}^2} = (+1)$, and $Q_{\overline{X}} = -Q_X$. Recall that E_8 is even with $(b^+, b^-) = (8, 0)$, and H is even with $(b^+, b^-) = (1, 1)$. Consequently, K3 has $(b^+, b^-) = (3, 19)$.

Problem 3. Compute the homology and cohomology groups of SO(3) with coefficients in $\mathbb{Z}$. For the result, verify both the universal coefficient theorem and Poincaré duality.

Hint Recall that SO(3) is homeomorphic to $\mathbb{RP}^3 = S^3/\mathbb{Z}_2$. Assume that we can compute homology via the cellular chain complex (see Example 2.42 in Hatcher's *Algebraic Topology*). The cellular chain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0,$$

with each $\mathbb{Z}$ in degree 3, 2, 1, and 0 respectively. Compute cohomology via the dual of this chain complex. (Don't forget that $\mathbb{RP}^n$ is orientable when n is odd.)

Problem 4. Let X be a connected, closed oriented 3-manifold with fundamental group $\pi_1(X, x_0)$. Using the universal coefficient theorem and Poincaré duality, compute the integral homology and cohomology groups of X in terms of $\pi_1(X, x_0)$. Use this to explain your answer to Problem 3.

Hint Recall that $H_1(X) \cong \pi_1(X, x_0)^{\mathrm{ab}}$, the abelianization of the fundamental group. Express your answer in terms of F and T, which denote the free and torsion parts of $\pi_1(X, x_0)^{\mathrm{ab}}$.

Problem 5. Using Čech cohomology, give a concise argument that the obstruction for reducing a principal $\mathrm{O}(k)$ bundle P to the subgroup $\mathrm{SO}(k)$ is given by an element $w_1 \in H^1(X; \mathbb{Z}_2)$. Give a formula for a Čech cocycle representing w_1 in terms of the transition functions for P. Finally, argue using the language of Čech cohomology that when $w_1 = 0$, the $\mathrm{SO}(k)$ reductions from a fixed P are parameterized by locally constant $\mathbb{Z}_2$-valued functions.

Remark. The twisted coefficients $\tilde{\mathbb{Z}}$ which appear in non-orientable Poincaré duality are $w_1 \times_{\rho_\pm} \mathbb{Z}$, where $\rho_\pm : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$ is multiplication by ± 1.