GAUGE THEORY AND DIFFERENTIAL INVARIANTS, WINTER 2015

Instructor: Dr. Ben Mares
Cycles: 2
Website: http://tensorial.com/math/gauge
Prerequisites: Nothing particular, but a broad background will help

1. Short Introduction

This course is essentially a repeat of the one I gave here at SISSA during Winter 2013. Based on feedback, this time I plan to spend a bit less time on background material and more time on applications.

This course will focus on applications of gauge theory to differential topology in four dimensions. The emphasis will be on the interplay between functional analysis and topology. This course consists of three major parts:

Background and context: Before addressing the main topics, I want to ensure that people have the context to understand the relevance of the results we discuss. Also I plan to review background material, in hopes of being broadly accessible.

Donaldson theory: This is the study of $SU(2)$ anti-self-dual instantons. Conceptually it is fairly simple, but the technicalities are difficult. I will examine some details while sketching the rest.

Seiberg-Witten theory: This is the theory of a $U(1)$ gauge field coupled to a spinor. Conceptually it is more complicated, but since the technicalities are vastly simpler, I will be able to provide a more rigorous treatment.

See Section 3 for a more detailed breakdown of each part.

Historically, topological field theory was instrumental to the discovery of Seiberg-Witten theory, but sadly the connection is beyond the scope of this course. In particular, I will not discuss the physical approach (supersymmetric path integrals). Nor will I discuss the algebraic geometric approach (semistable sheaves).

Since Donaldson theory was mostly replaced by Seiberg-Witten theory, one might question why I should present it. Not only is it historically interesting, but the past few years have seen a resurgence in Donaldson theory.

2. Introduction

Kevin Iga [1] nicely summarizes the development of gauge theory as a tool in four-dimensional differential topology:

Date: January 26, 2015.

> In 1983, Donaldson shocked the topology world by using instantons from physics to prove new theorems about four-dimensional manifolds, and he developed new topological invariants. In 1988, Witten showed how these invariants could be obtained by correlation functions for a twisted $\mathcal{N} = 2$ SUSY gauge theory. In 1994, Seiberg and Witten discovered dualities for such theories, and in particular, developed a new way of looking at four-dimensional manifolds that turns out to be easier, and is conjectured to be equivalent to, Donaldson theory.

In conjunction with the results of Freedman and Taubes, Donaldson theory provided:

- Large classes of topological four-manifolds which admit no smooth structure
- Examples of "exotic" topological four-manifolds which admit multiple smooth structures, including $\mathbb{R}^4$
- Invariants which are sometimes capable of distinguishing smooth structures

Due to technical obstacles, the proofs from Donaldson theory were quite cumbersome. Upon the discovery of Seiberg-Witten theory, four-dimensional differential topology was revitalized with simpler proofs, simpler invariants, and new theorems. When the four-manifold is symplectic, Taubes has shown that the Seiberg-Witten invariant counts the number of pseudo-holomorphic curves for a generic almost-complex structure.

One particular application I plan to highlight was one of the first major triumphs of Seiberg-Witten theory. The *minimal genus problem* asks: given an integral homology class of degree two inside a smooth four-manifold, what is the minimal genus of a smoothly embedded Riemann surface Σ which represents it? Some simpler homology classes might be representable by smoothly embedded spheres or tori, while more complicated classes may require surfaces of higher genus.

In the case when our four-manifold is a complex manifold and Σ is a complex submanifold, the genus $g(\Sigma)$ is determined by the *adjunction formula*

$$2\,g(\Sigma) - 2 = \langle K, [\Sigma] \rangle + [\Sigma] \cdot [\Sigma] \,.$$

For example on $\mathbb{CP}^2$, $K = -3\,[H]$, $[\Sigma] = d\,[H]$, and $[H] \cdot [H] = 1$, where d is the degree of Σ. This yields the classical formula $g(\Sigma) = \frac{1}{2}(d-1)(d-2)$. The *Thom conjecture* asserts that inside $\mathbb{CP}^2$, the genus does not decrease if we allow smooth (but non-algebraic) Σ. Kronheimer and Mrowka solved the Thom conjecture by discovering a simple proof based on Seiberg-Witten theory. Their idea yields a beautiful extension of the adjunction formula to an *adjunction inequality* for non-algebraic manifolds, where the Seiberg-Witten invariants produce lower bounds for $g(\Sigma)$.

3. Lecture plan (tentative)

- Background and context
  - Classification of manifolds, $n \neq 4$
    - Equivalences: homology, homotopy, homeomorphism, diffeomorphism
    - Requirements: oriented, simply-connected, homotopy sphere
    - Dimensions: 0, 1, 2, 3, ≥ 5
    - Poincaré duality and Intersection forms
    - Fake, non-smoothable, and exotic manifolds
  - Classification of four-manifolds
    - Freedman's classification of topological four-manifolds
    - Donaldson's Theorem

  - Hodge de Rham Theory
    * Sobolev spaces
    * Hodge decomposition theorem
    * Resolving the constant sheaf, and the de Rham theorem
    * "Algebraic topological" vs. "differential topological"
  - Bundle theory
    * Čech cohomology
    * Topological line bundles and homological algebra
    * Non-abelian Čech cohomology
    * Classifying space theory
    * Dold-Whitney theorem
    * Principal bundle formalism, associated bundles
    * The adjoint bundle, and connections
- Donaldson theory
  - Rotations in four dimensions, and self-duality
  - The anti-self-duality equation, and relation to Yang-Mills
  - The deformation complex
  - Banach manifolds and the inverse function theorem
  - Gauge fixing and local models
  - The parameterized moduli space and generic metrics
  - Reducible connections
  - Taubes' grafting procedure
  - Uhlenbeck compactness
  - Donaldson's Theorem
  - Donaldson invariants
- Seiberg-Witten theory
  - Spin geometry
    * Spin groups
    * Representation theory
    * Lifting to spin structures
    * Spinor representations
    * Clifford multiplication and the Dirac operator
    * Spin^c representations
    * Lifting to spin^c structures
    * Spin^c connections and Dirac operators
  - The Seiberg-Witten equations
    * The deformation complex
    * Why they are so much easier
  - Compactness
    * Review of Riemannian geometry
    * Weitzenböck formula
    * Compactness estimate
  - Seiberg-Witten proof of Donaldson's theorem
  - Thom conjecture, and the minimal genus of embedded surfaces

References

[1] Iga, K., *What do topologists want from Seiberg-Witten Theory?* arXiv:hep-th/0207271