# Homological view of intersections

The "intersection form" on cohomology derives its name from the way it counts intersection numbers of the corresponding Poincaré dual submanifolds. The wedge product is Poincaré dual to intersection. Let's give a brief overview of this perspective.

Suppose we have an oriented embedded submanifold $M \subset X$ of dimension $i$. In local coordinates $x^1, \ldots, x^n$ on $X$, $M$ can be put into standard form as $0 = x^{i+1} = x^{i+2} = \cdots = x^n$. The orientation on $M$ is represented by $dx^1 \wedge \cdots \wedge dx^i \in \Omega^i(M)$.

Suppose $X$ is also oriented so that $dx^1 \wedge \cdots \wedge dx^n$ is positive. We wish to find the Poincaré dual to the fundamental class $[M]$, represented by some differential form $\alpha \in \Omega^{n-i}(X)$. There is a construction of Mathai and Quillen for such an $\alpha$ with its support localized around $M$ (the Mathai-Quillen representative for the Thom class of the normal bundle). In particular, suppose $\rho(x^{i+1}, \ldots, x^n)$ is a bump function of the normal coordinates with unit integral $\int \rho(x^{i+1}, \ldots, x^n)\, dx^{i+1} \cdots dx^n = 1$ in the normal directions. Then locally, the constructed representative looks like $\beta = \rho(x^{i+1}, \ldots, x^n)\, dx^{i+1} \wedge \cdots \wedge dx^n$. While homology classes look like oriented submanifolds (but allowing for singularities), cohomology classes appear localized around *co*-oriented submanifolds, meaning that the normal bundle (rather than tangent bundle) is oriented. Poincaré duality effectively uses the orientation of $X$ to transform the orientation of $M$ into a co-orientation of $M$. Any two of these determine the third at any point (after fixing conventions, which we will skip):

$$\begin{aligned} \text{orientation on } M :& dx^1 \wedge \cdots \wedge dx^i \\ \text{co-orientation on } M :& dx^{i+1} \wedge \cdots \wedge dx^n \\ \text{orientation on } X :& dx^1 \wedge \cdots \wedge dx^n \end{aligned}$$

Now suppose that $N \subset X$ is another submanifold of dimension $j$ with transverse intersection $M \pitchfork N$. In local coordinates around the intersection, $N$ looks like $0 = x^1 = \cdots = x^{n-j}$ with orientation $dx^{n-j+1} \wedge \cdots \wedge dx^n$. The corresponding differential form looks like $\alpha = \tilde{\rho}(x^1, \ldots, x^{n-j})\, dx^1 \wedge \cdots \wedge dx^{n-j}$.

The intersection $M \pitchfork N$ is $0 = x^{i+1} = \cdots = x^{n-j}$ of dimension $n - i - j$. (Note that the intersection is empty whenever $i + j < n$, since the manifolds can be perturbed off each other.) We check that the wedge product gives a good representative on the intersection:

$$\alpha \wedge \beta = \tilde{\rho}(x^1, \ldots, x^{n-j})\rho(x^{i+1}, \ldots, x^n)\, dx^1 \wedge \cdots \wedge dx^{n-j} \wedge dx^{i+1} \wedge \cdots \wedge dx^n.$$

Observe that this clearly co-orients the intersection, and integrates to one in the normal directions.

If $M$ and $N$ have complementary dimensions so that $n - i - j = 0$, then the intersection is a union of discrete points. The wedge product $\alpha \wedge \beta \in \Omega^n(X)$ co-orients each point. Furthermore, around each point, every direction is normal. Thus $\int_X \alpha \wedge \beta$ localizes to the intersection points, with each intersection point contributing $\pm 1$, depending on orientation.

Note that this algebraic intersection count is independent of perturbations of $M$ and $N$. More generally, the intersection count makes sense at the level of singular chains and cochains, and thus is independent up to homology equivalence. For instance, if $M$ is the boundary of an oriented compact manifold $\tilde{M}$, and if the embedding $M \hookrightarrow X$ extends to a continuous map $\tilde{M} \to X$, then the fundamental class $[M] = [\partial \tilde{M}] = 0$, so $M$ is homologically trivial, and the algebraic intersection of $M$ and $N$ will always vanish.

We obtain the following geometric interpretation of the intersection form. Consider a closed oriented 4-manifold $X$ with intersection form $Q$. Choose a basis $\{[\alpha_i]\}_{i=1}^{b^2(X)}$ of $H^2_{\text{free}}(X;\mathbb{Z})$. Represent $\text{PD}([\alpha_i])$ by transverse oriented embedded surfaces $\{[\Sigma_i]\}_{i=1}^{b^2(X)}$. In this basis, the off-diagonal matrix elements $Q_{ij}$ give the algebraic count $\#(\Sigma_i \cap \Sigma_j)$. The diagonal element $Q_{ii}$ is the self-intersection number of $\Sigma_i$. Specifically, consider a tiny deformation from $\Sigma_i$ to another surface $\Sigma_i'$ which is transverse to $\Sigma_i$. Then $Q_{ii} = \#(\Sigma_i \cap \Sigma_i')$. One way to specify such a tiny deformation is to give a section of the normal bundle $N(\Sigma_i)$. The normal bundle of $\Sigma_i$ can be identified diffeomorphically with a "tubular neighborhood" of $\Sigma_i$, with $\Sigma_i$ corresponding to the zero section. Thus $Q_{ii}$ is the Euler characteristic of the normal bundle of $\Sigma_i$:

$$Q_{ii} = \#(\Sigma_i \cap \Sigma_i') = \#\text{zeroes of generic section of } N(\Sigma_i) = \chi(N(\Sigma_i)).$$

## Classification of unimodular symmetric bilinear forms

A unimodular bilinear form $Q$ is *positive/negative definite* if $Q(x,x)$ is always positive/negative for nonzero $x$. If $Q$ is neither positive definite nor negative definite, then $Q$ is called *indefinite*. If $Q(x,x)$ is always even, then $Q$ is called *even*. Otherwise, $Q$ is called *odd*. For example,

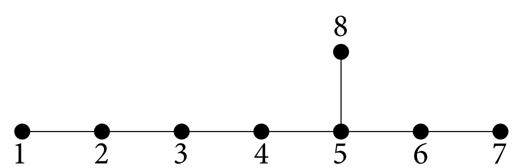
$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is even and indefinite, since

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2x_1x_2.$$

Note that $Q$ is even iff its diagonal elements are always even. (This is because off-diagonal entries automatically acquire a factor of two.)

An even positive-definite form arises via the Cartan matrix for the Lie algebra $E_8$:




$$E_8 := \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{pmatrix}.$$

Classification of unimodular definite forms is not understood, and the numbers grow rapidly with rank. Thankfully we are saved from this hopeless situation by

**Theorem** (Donaldson). *If $X$ is a simply-connected 4-manifold with $Q_X$ definite, then*

$$Q_X \sim \pm \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} = m\,(\pm 1)\,.$$

Indefinite forms are much easier to classify. If $Q$ is indefinite and odd, then $Q$ is diagonal:

$$Q \sim m\,(+1) \oplus n(-1),$$

with $m, n > 0$. If $Q$ is indefinite and even, then

$$Q \sim \pm m\, E_8 \oplus n\, H,$$

where $n > 0$, $m \geq 0$, and $-E_8$ is to be understood as $E_8$ with the opposite sign.

We can summarize these results with the following table:

| | odd | even |
|---|---|---|
| indefinite | $m\,(+1) \oplus n(-1)$ | $\pm m\, E_8 \oplus n\, H$ |
| definite | too difficult, but we only encounter $m\,(\pm 1)$ | |

These exhaust all possibilities for intersection forms!

*Remark.* Indefinite forms are completely classified by rank, signature, and type (even/odd). For example, $E_8 \oplus -E_8 \sim 8H$ since it is even of signature zero. Also, $E_8 \oplus (-1) = 8\,(+1) \oplus (-1)$ which is odd of signature 7.

Now that we understand the possible cohomology of closed oriented 4-manifolds, we can try and ascend our classification hierarchy to understand smooth 4-manifolds. Recall that we restrict to *simply-connected* closed 4-manifolds because the general classification problem would encompass the impossible classification of all finitely presented groups, which arise as fundamental groups. Now we attempt to use cohomology to ascend the classification hierarchy. The first step proceeds without difficulty. Hatcher gives a complete proof of:

**Proposition** (Algebraic Topology, 4C.3). *For a simply-connected closed topological 4-manifold, cohomology determines homotopy type.*

Thanks to the incredible work of Freedman, we can ascend to homeomorphism classification:

**Theorem** (Freedman). *For any unimodular symmetric bilinear form $Q$, there is a closed simply-connected topological 4-manifold with $Q$ as its intersection form. Furthermore,*

- *if $Q$ is even, the manifold is unique up to homeomorphism,*
- *if $Q$ is odd, there are two homeomorphism classes, at least one of which is not smoothable.*

Note that for any intersection form $Q$, there is at most one homeomorphism class containing a smooth manifold. Consequently, two simply-connected smooth 4-manifolds $X_1$ and $X_2$ are homeomorphic iff $Q_{X_1} \sim Q_{X_2}$!