Universal coefficients and Poincaré duality (continued)

Recall the universal coefficient theorem

$$0 \to \operatorname{Ext}(H_{i-1}(X;\mathbb{Z}), A) \to H^i(X;A) \to \operatorname{Hom}(H_i(X;\mathbb{Z}), A) \to 0,$$

which is split, so there is an isomorphism $H^i(X;A) \cong \operatorname{Ext}(\cdots) \oplus \operatorname{Hom}(\cdots)$. By classification of finitely generated abelian groups, when X is closed,

$$H_i(X;\mathbb{Z}) \cong \mathbb{Z}^{b_i(X)} \oplus T_i(X),$$

for some integer $b_i(X)$ and some torsion subgroup $T_i(X) \cong \mathbb{Z}_{i_1} \oplus \cdots \oplus \mathbb{Z}_{i_k}$.

To compute real cohomology

$$H^i(X;\mathbb{R}) \cong \operatorname{Ext}(\cdots) \oplus \operatorname{Hom}(\cdots),$$

we get $\operatorname{Ext}(H_{i-1}(X;\mathbb{Z}), \mathbb{R}) = 0$ and $\operatorname{Hom}(H_i(X;\mathbb{Z}), \mathbb{R}) \cong \mathbb{R}^{b_i(X)}$, thus

$$H^i(X;\mathbb{R}) \cong \mathbb{R}^{b_i(X)},$$

and we see that $b_i(X) = b^i(X)$, where $b^i(X)$ are the familiar Betti numbers from de Rham cohomology. For integer cohomology, we compute $\operatorname{Ext}(H_{i-1}(X;\mathbb{Z}), \mathbb{Z}) = T_{i-1}$ and $\operatorname{Hom}(H_i(X;\mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}^{b^i(X)}$, so

$$H^i(X;\mathbb{Z}) \cong \mathbb{Z}^{b^i(X)} \oplus T_{i-1}.$$

We now be more precise regarding the relation between real and integer cohomology for compact manifolds. We have natural isomorphisms

$$H^i(X;\mathbb{R}) = \operatorname{Hom}(H_i(X;\mathbb{Z}), \mathbb{R}) = \operatorname{Hom}(H_i(X;\mathbb{Z}), \mathbb{Z}) \otimes \mathbb{R} = H^i_{\text{free}}(X;\mathbb{Z}) \otimes \mathbb{R},$$

where we use the fact that torsion disappears under tensor or hom with $\mathbb{R}$. Thus, we may view $H^i_{\text{free}}(X;\mathbb{Z})$ as an integer lattice inside of the vector space $H^i(X;\mathbb{R})$.

Poincaré duality is a different identification of homology with cohomology, giving an isomorphism

$$H^i_c(X;\mathbb{Z}) \to H_{n-i}(X;\tilde{\mathbb{Z}}),$$

where $\tilde{\mathbb{Z}}$ denotes homology with "twisted coefficients." If X is closed, then $H^k_c(X;A) = H^k(X;A)$. If X is oriented, then $\tilde{\mathbb{Z}} = \mathbb{Z}$, and consequently,

$$H^i(X;\mathbb{Z}) \cong \mathbb{Z}^{b^{n-i}(X)} \oplus T_{n-i}(X).$$

Thus

$$T_i(X) := T(H_i(X;\mathbb{Z})) \overset{\text{UC}}{\cong} T(H^{i+1}(X;\mathbb{Z})) \overset{\text{PD}}{\cong} T(H_{n-i-1}(X)) =: T_{n-i-1}(X).$$

Note that since $H_0(X;\mathbb{Z}) = \mathbb{Z}^{\#\text{components}(X)}$ is free, $T_0 = 0$. Therefore, $T_{n-1} \cong T_0 = 0$, and $T_n \cong T_{-1} = 0$.

Now the homology and cohomology groups are very tightly constrained. For example, for a connected, compact oriented 4-manifold, by the universal coefficient theorem,

i	0	1	2	3	4
$H_i(X;\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}^{b^1(X)} \oplus T_1$	$\mathbb{Z}^{b^2(X)} \oplus T_2$	$\mathbb{Z}^{b^3(X)} \oplus T_3$	$\mathbb{Z}^{b^4(X)} \oplus T_4$
$H^i(X;\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}^{b^1(X)}$	$\mathbb{Z}^{b^2(X)} \oplus T_1$	$\mathbb{Z}^{b^3(X)} \oplus T_2$	$\mathbb{Z}^{b^4(X)} \oplus T_3$

and then by Poincaré duality,

i	0	1	2	3	4
$H_i(X;\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}^{b^1(X)} \oplus T_1$	$\mathbb{Z}^{b^2(X)} \oplus T_1$	$\mathbb{Z}^{b^1(X)}$	$\mathbb{Z}$
$H^i(X;\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}^{b^1(X)}$	$\mathbb{Z}^{b^2(X)} \oplus T_1$	$\mathbb{Z}^{b^1(X)} \oplus T_1$	$\mathbb{Z}$

For the Poincaré homology sphere P, since $H_1(X;\mathbb{Z}) \cong \pi_1^{\text{ab}}(P, x_0) = \tilde{I}^{\text{ab}} = 0$, one easily computes

i	0	1	2	3
$H_i(X;\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}$
$H^i(X;\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}$

Since there are no possibilities for nontrivial cup products, in both homology and cohomology, P looks exactly like S^3, even in the ring structure of cohomology.

Exercise. Compute the homology and cohomology groups of a connected closed oriented 3-manifold X in terms of $\pi_1(X)$.

In the compact case, an orientation is equivalent to a choice of generator of $H_i(X;\mathbb{Z}) \cong \mathbb{Z}$, known as the fundamental class $[X]$. This homology class should be thought of as a (oriented) triangulation of X. Concretely, a nowhere vanishing element of $\omega \in \Omega^n(X)$ determines a positive atlas, in which ω is positive in each chart. Thus $\int_X \omega > 0$. Clearly $\omega \in \ker d$, since $\Omega^{n+1}(X) = 0$. But $[\omega] \neq 0 \in H^n(X;\mathbb{R})$ since by Stokes' theorem,

$$[\omega] = 0 \implies \omega = d\eta \implies \int_X \omega = \int_X d\eta = \int_{\partial X = \varnothing} \eta = 0.$$

Thus ω must generate the one-dimensional vector space $H^n(X;\mathbb{R})$. After rescaling ω by the appropriate positive constant, ω determines a generator for $H^n(X;\mathbb{Z}) \subset H^n(X;\mathbb{R})$. By the universal coefficient theorem, $H_n(X;\mathbb{Z})$ is the dual lattice inside the dual vector space $H_n(X;\mathbb{R})$, and we have the corresponding dual basis element $[X] \in H_n(X;\mathbb{Z})$.

In the compact oriented case, the Poincaré duality isomorphism $H^i(X;\mathbb{Z}) \to H_{n-i}(X;\mathbb{Z})$ is *cap product* with the fundamental class $[X]$:

$$a \mapsto [X] \frown a.$$

Specifically, for any ring R, the cap product can be defined as partial evaluation $H_{i+j}(X;R) \times H^i(X;R) \to H_j(X;R)$. Ignoring torsion, the cap product is dual to the cup product. If we view $H^{\text{free}}_{n-i}(X;\mathbb{Z})$ as the dual group to $H^{n-i}_{\text{free}}(X;\mathbb{Z})$, then the map induced by the cap product and universal coefficients

$$H^i_{\text{free}}(X;\mathbb{Z}) \to H^{\text{free}}_{n-i}(X;\mathbb{Z}) \xrightarrow{\cong} H^{n-i}_{\text{free}}(X;\mathbb{Z})^*$$

corresponds to the "cup product and integrate" map

$$H^i(X;\mathbb{R}) \to H^{n-i}(X;\mathbb{R})^*$$
$$a \mapsto \left(b \mapsto \int_X (a \smile b)\right)$$

which we used to define the intersection form via de Rham cohomology. This gives the intersection form the structure of a unimodular integer bilinear form, as was previously claimed.

Representing homology classes via submanifolds

Homology classes are represented by simplicial "cycles" i.e. chains without boundary. The "Steenrod problem" asks whether a class $a \in H_p(X;\mathbb{Z})$ can be represented by a manifold. Specifically, is there a closed oriented smooth manifold M and a continuous map $f : M \to X$ such that the image of a fundamental class $f_*([M]) = a$? In his work on cobordism, Thom showed that this is not always possible. However, there is always an integer multiple of a which is representable. This question can be strengthened to require that f be either an immersion (locally an embedding, but globally there can be self-intersections) or an embedding. In particular, when we discuss the minimal genus problem, we want to know that classes $a \in H_2(X;\mathbb{Z})$ are representable by embedded submanifolds. Assuming some homotopy theory, we can prove via Poincaré duality that this is always possible when X is a closed oriented 4-manifold.

To warm up, consider X closed and oriented, and $a \in H_{n-1}(X;\mathbb{Z})$. We can write any such a as the Poincaré dual $a = \mathrm{PD}(\alpha)$ to $\alpha \in H^1(X;\mathbb{Z})$. The homotopy theory fact we require is that cohomology groups are *representable*, i.e. $H^p(X;A)$ is in bijection with homotopy classes of maps from X into some space $K(A,p)$, called an Eilenberg-MacLane space.

$$H^p(X;A) \overset{\text{bij}}{\cong} [X, K(A,p)].$$

In particular,

$$H^1(X;\mathbb{Z}) \overset{\text{bij}}{\cong} [X, K(\mathbb{Z},1)],$$

and $K(\mathbb{Z},1) = S^1$. Choosing a representative map $h : X \to S^1$, the corresponding cohomology class is given by pullback of the generator $\xi = [d\theta/2\pi] \in H^1(S^1;\mathbb{Z})$. Thus each cohomology class $\alpha \in H^1(X;\mathbb{Z})$ is $\alpha = h^*(\xi)$ for some map $h : X \to S^1$. We can represent the homology class $\mathrm{PD}(\xi)$ by a point pt, so that $\mathrm{PD}(\xi) = [\mathrm{pt}] \in H^0(S^1)$.

Poincaré duality is functorial in the sense that

$$\mathrm{PD}(\alpha) = \mathrm{PD}(h^*(\xi)) = h^*(\mathrm{PD}(\xi)) = h^*([\mathrm{pt}]) = \left[h^{-1}(\mathrm{pt})\right],$$

where h^{-1} denotes the inverse image, assuming that h is transverse to the given representative pt of $\mathrm{PD}(\xi)$.

In summary, to represent a homology class $a \in H_{n-1}(X;\mathbb{Z})$, write $a = \mathrm{PD}(\alpha)$, and then choose a map $h : X \to K(\mathbb{Z},1)$ such that $\alpha = h^*(\xi) \in H^1(X;\mathbb{Z})$, where ξ is the generator in $\mathbb{Z} = H^1(K(\mathbb{Z},1);\mathbb{Z})$. Since $K(\mathbb{Z},1)$ happens to be a manifold, we can look for an explicit codimension 1 submanifold $Y \subset K(\mathbb{Z},1)$ corresponding to $\mathrm{PD}(\xi) \in H_1(K(\mathbb{Z},1);\mathbb{Z})$. (Here, Y happens to be a point.) After perturbing h to make it transverse to Y, a is the fundamental class of the preimage $h^{-1}(Y)$.

With slight modification, the same argument carries through for $a \in H_{n-2}(X;\mathbb{Z})$. In this case, $K(\mathbb{Z},2) = \mathbb{CP}^\infty$, where $\mathbb{CP}^\infty$ is the union of the inclusions $\mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \cdots \subset \bigcup_i \mathbb{CP}^i = \mathbb{CP}^\infty$. While this is not a manifold in the traditional sense, the "cellular approximation theorem" of homotopy theory allows us to homotope h to some $\mathbb{CP}^N$ with N finite. In this case, for ξ the generator of $H^2(\mathbb{CP}^N;\mathbb{Z}) \cong \mathbb{Z}$, the Poincaré dual $\mathrm{PD}(\xi) = \left[\mathbb{CP}^{N-1}\right]$. Thus when h is transverse to $\mathbb{CP}^{N-1}$, we obtain a suitable embedded submanifold $h^{-1}(\mathbb{CP}^{N-1})$ whose fundamental class represents a.