Classification of manifolds

Now let's examine classification of manifolds. In dimensions three and below, homeomorphims classes and diffeomorphism classes agree, so we sloppily refer to "isomorphism" of manifolds to avoid the distinction.

We might as well start in dimension zero, where a manifold is by definition a (countable) collection of discrete points. Any manifold is a disjoint union of its connected components, so it makes sense to study only connected manifolds. The only connected 0-manifold is a point.

Connected 1-manifolds are diffeomorphic to either $\mathbb{R}$ or S^1. This is a good point to mention the notion of a *manifold with boundary*.

Definition. A *manifold with boundary* is a manifold which is locally isomorphic to "relatively open" subsets of the closed half-plane $\{\vec{x} \in \mathbb{R}^n | x^1 \geq 0\}$. Points of the manifold corresponding in some (any) chart to points with $x^1 = 0$ are called *boundary points*.

Remark. A profound statement which somewhat underlies the foundation of homology theory is this: given a manifold with boundary, its boundary is a manifold without boundary. Symbolically, $\partial^2 = 0$, where ∂ is the operator which takes a manifold and gives its boundary. The relation $d^2 = 0$ is the dual statement under Stokes' theorem.

There are a total of four connected 1-manifolds with boundary.

	compact	noncompact
empty boundary	S^1	$\mathbb{R}$
nonempty boundary	$[0,1]$	$[0,\infty)$

(Note that $(0,1] \cong [0,\infty)$ by the diffeomorphism $x \mapsto x^{-1} - x$.)

In dimension 2 (surfaces) can be quite nasty in general. (Consider for example $\mathbb{R}^2$ – cantor set.) The situation becomes much nicer if we restrict to compact manifolds. Compact surfaces with boundary must have boundary which is compact with empty boundary, i.e. finitely many copies of S^1. For simplicity, we consider only surfaces without boundary.

For example, we have S^2, the torus $T^2 = T$, as well as our first examples of non-orientable surfaces $S^2/\text{antipodal} = \mathbb{RP}^2 = P$, and the Klein bottle K.

Using the operation of connected sum, we can form a composite surface from two new ones. This induces an abelian monoid (=group without inverse axiom) structure on isomorphism classes of surfaces. We obtain the relations

$$S^2 \# X = X, \quad \forall X,$$
$$P \# P = K,$$
$$P \# P \# P = K \# P = T \# P.$$

Remark. S^2 is the identity of the monoid.

Remark. These generators and relations are complete, i.e. the resulting monoid is isomorphic to the monoid of isomorphism classes of connected compact surfaces.

Remark. The monoid is generated by T and P (the second relation eliminates K). Given a word in T and P, if P appears, then by the last relation we can trade T for P^2.

Thus the isomorphism classes correspond to the orientable surfaces

$$\Sigma_g := T^{\# g}, \quad g \geq 0, \quad \left(\Sigma_0 := S^2\right)$$

plus the non-orientable surfaces

$$P^{\# k}, \quad k > 0.$$

From here, we would want to show two things:

- every compact connected 2-manifold is isomorphic to one of these examples, and
- these examples are distinct.

There are various ways to prove the first statement, but they all tend to be fairly combinatorial, so they are of little interest to us. Furthermore, the corresponding statement in four dimensions is hopeless, since there is no conjectured enumeration of four-manifolds.

The second statement is far more interesting for our purposes. Assuming that every 2-manifold has a triangulation, we can compute the Euler characteristic $\chi(X) = V - E + F$ as

$$\chi(P^{\# k}) = 2 - k, \quad \chi(\Sigma_g) = 2 - 2g.$$

Assuming the classification, we observe that the pair consisting of $\{\text{orientability}(X), \chi(X)\}$ is a complete invariant, meaning that two manifolds are isomorphic iff they have the same such invariants.

At this point, it is instructive to bring in the notion of de Rham cohomology for smooth manifolds. Recall the exact sequence

$$\Omega^0(\mathbb{R}^n) \xrightarrow{d} \Omega^1(\mathbb{R}^n) \xrightarrow{d} \Omega^2(\mathbb{R}^n) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(\mathbb{R}^n) \to 0.$$

Specifically, if $\omega \in \Omega^0(\mathbb{R}^n)$ and $d\omega = 0$, then ω is a locally constant function. But if $\omega \in \Omega^p(\mathbb{R}^n)$ with $p > 0$, then $d\omega = 0 \implies \omega = d\eta$ for some $\eta \in \Omega^{p-1}(\mathbb{R}^n)$. Replacing $\mathbb{R}^n$ by X, the sequence no longer needs to be exact:

$$\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(X) \to 0.$$

It still satisfies $d^2 = 0$, which is a local computation, i.e. ker $\supset$ image. Such a sequence is called a *chain complex*. What goes wrong is that if $d\omega = 0$ for $\omega \in \Omega^{p>0}(X)$, then it's not necessarily true that $\omega = d\eta$. Since locally X looks like $\mathbb{R}^n$, we can always locally find some η. But there is no reason that these local solutions should patch compatibly to form a global solution.

We first take a tautological approach to this issue.

$$d\omega = 0 \implies \omega \in \ker\left(\Omega^p(X) \xrightarrow{d} \Omega^{p+1}(X)\right).$$

$$\omega = d\eta \implies \omega \in \text{image}\left(\Omega^{p-1}(X) \xrightarrow{d} \Omega^p(X)\right).$$

The obstruction to finding a global solution is the quotient space

$$[\omega] \in H^p(X) = \frac{\ker\left(\Omega^p(X) \xrightarrow{d} \Omega^{p+1}(X)\right)}{\text{image}\left(\Omega^{p-1}(X) \xrightarrow{d} \Omega^p(X)\right)},$$

so that $\omega = d\eta$ iff $d\omega = 0$ (local obstruction) and then $[\omega] = 0$ (global obstruction).

The vector space H^p has many important properties. If X is a compact manifold, then $H^p(X)$ is a finite-dimensional vector space.

Remark. It's rarely possible to compute $H^p(X)$ directly. Instead, one uses techniques from homological algebra, namely the Mayer-Vietoris sequence. The book by Bott and Tu is a superb reference for this subject.

Definition. For $p = 0, \dots, n$, the p-th *Betti number* of a manifold X is $b^p(X) := \dim H^p(X)$.

The Betti numbers satisfy many nice properties:

- $b^0(X) =$ #components. Thus if X is connected, then $b^0(X) = 1$.
- The Euler characteristic is the alternating sum $\chi = b^0 - b^1 + b^2 - \cdots$.
- (Poincaré duality) If X is an oriented compact n-manifold, then $b^p = b^{n-p}$.
- If X is a connected compact n-manifold, then

$$b^n(X) = \begin{cases} 1 & \text{if } X \text{is orientable,} \\ 0 & \text{if } X \text{is non-orientable.} \end{cases}$$

Using these properties, and our previous statement regarding the Euler characteristics, we see that for connected compact surfaces,

$$\begin{aligned} b^0(P^{\#k}) &= 1, & b^0(\Sigma_g) &= 1, \\ b^2(P^{\#k}) &= 0, & b^2(\Sigma_g) &= 1, \\ b^1(P^{\#k}) &= k-1, & b^1(\Sigma_g) &= 2g. \end{aligned}$$

In particular, $\{b^1, b^2\}$ form a complete set of invariants, equivalent to our previous choice $\{\text{orientability}(X), \chi(X)\}$.

We now focus on the orientable case. Note that b^1 is always even. We can explain this via a slight refinement of Poincaré duality. But first an exercise:

Exercise. Verify that the wedge product on $\Omega^\bullet(X)$ induces a well-defined product on $H^\bullet(X)$, called the *cup product* $\smile$, i.e. $[\omega_1] \smile [\omega_2] := [\omega_1 \wedge \omega_2]$.

This endows $H^\bullet(X)$ with the structure of a skew-commutative graded ring, meaning that for $a \in H^p(X)$, $b \in H^q(X)$,

$$a \smile b = (-1)^{pq}\, b \smile a \in H^{p+q}(X).$$

We get the following refinement of Poincaré duality:

Theorem. *If X is a compact, oriented, connected n-manifold, then for each p, the cup product is a nondegenerate bilinear map*

$$H^p(X) \times H^{n-p}(X) \to H^n(X) \cong \mathbb{R}.$$

Specifically, if we use the natural identification of $H^n(X)$ with $\mathbb{R}$, then the cup product induces a map

$$\begin{aligned} H^p(X) &\to \left(H^{n-p}(X)\right)^* \\ a &\mapsto (b \mapsto a \smile b \in \mathbb{R}). \end{aligned}$$

Definition. A bilinear map is *nondegenerate* when this map is an isomorphism.

To avoid repitition, we will always assume that X is connected, compact, and oriented of dimension n.

Our previous notion of Poincaré duality follows from

$$b^p = \dim H^p(X) = \dim \left(H^{n-p}(X)\right)^* = b^{n-p}.$$

This enhanced version of Poincaré duality also detects certain intrinsic constraints on the cup product structure of $H^\bullet(X)$. For example, if $n/2$ is an odd integer, i.e. $n = 2, 6, 10, \ldots$, then the cup product

$$H^{n/2}(X) \times H^{n/2}(X) \to \mathbb{R}$$

is antisymmetric! Choosing any basis of $H^{n/2}(X)$, we obtain a nondegenerate antisymmetric matrix. Thus the eigenvalues are nonzero, purely imaginary, and come in conjugate pairs, so $b^{n/2}(X)$ is even. Using the antisymmetric version of Graham-Schmidt, it is possible to choose a "symplectic basis" such that the matrix takes the form

$$\begin{pmatrix} & -1 & & & \\ 1 & & & & \\ & & & -1 & \\ & & 1 & & \\ & & & & \ddots \end{pmatrix}.$$

The operation of connected sum amounts to a direct sum (block diagonal composition) on the level of intersection forms. The intersection form of T is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so the intersection form of $\Sigma_g = \underbrace{T\#\cdots\#T}_{g}$ is

$$\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{g}.$$

In this way, the structure of Σ_g is reflected in its cohomology.

If $n/2$ is an even integer, i.e. $n = 4, 8, 12, \ldots$, then

$$H^{n/2}(X) \times H^{n/2}(X) \to \mathbb{R}$$

is symmetric, so the eigenvalues are real and nonzero. We can choose a basis so that the matrix is

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix},$$

and we get two new invariants $b^+(X)$ = #positive eigenvalues and $b^-(X)$ = #negative eigenvalues which satisfy $b^+ + b^- = b^2$. The combination $\sigma(X) = b^+(X) - b^-(X)$ is called the *signature*.

Remark. Technically, it's wrong to talk about the eigenvalues of a bilinear form, since a bilinear form is not an endomorphism. The transformation law is different. (Under a change of basis, the matrix of a bilinear form transforms as $\mathbf{Q} \mapsto \mathbf{G}^T\mathbf{Q}\mathbf{G}$, while an endomorphism transforms as $\mathbf{L} \mapsto \mathbf{G}^{-1}\mathbf{L}\mathbf{G}$.) The actual eigenvalues depend on the choice of basis, but the number of positive eigenvalues of any matrix representing a bilinear form gives the maximal dimension of any positive-definite subspace.

Later when we define integer cohomology, we will see that this intersection matrix has integer entries, is well-defined up to an integral change of basis, and is *unimodular*, i.e. the determinant of its matrix is ± 1.

Remark. An integer matrix is invertible over the integers iff the determinant is ± 1. This can be seen explicitly via the formula $A^{-1} = \operatorname{adj}(A)/\det A$, where $\operatorname{adj}(A)$ is the transpose of the cofactor matrix (no division). It is natural to denote such matrices by $\mathrm{GL}(n;\mathbb{Z})$, but not in this particular context. Here it is more natural to interpret the "invertibility" as the condition that the duality map $H^{n/2}(X;\mathbb{Z}) \to (H^{n/2}(X;\mathbb{Z}))^*$ is an isomorphism of $\mathbb{Z}$-modules.

Classification of such integral bilinear forms is a rich number-theoretic subject.

Theorem (Freedman's classification of topological four-manifolds)**.** *Every equivalence class of integral unimodular (*$\det = \pm 1$*) symmetric bilinear form corresponds to either 1 or 2 homeomorphism classes of* simply connected *compact topological 4-manifolds. (It corresponds to 1 such homeomorphism class iff the diagonal entries of the matrix are all even.)*