Motivation

Our proof of the Hodge Theorem relied on two unproven lemmas: regularity ($\alpha \in \Omega^p(X)$, $\Delta\omega = \alpha \implies \omega \in \Omega^p(X)$) and compactness ($\|\omega_i\|_{L^2}$ and $\|\Delta\omega_i\|_{L^2}$ both bounded $\implies$ $\{\omega_i\}$ has a convergent subsequence).

For gauge theory, we will routinely use the Hodge decomposition for elliptic operators between Sobolev spaces. Thus we will outline a theory of Sobolev spaces and elliptic operators over closed manifolds, which makes these lemmas obvious. It will be based on the analysis of L^2 functions on the n-torus T^n.

Let X be a closed oriented Riemannian manifold of dimension n.

Recall that a *Banach space* is a complete normed vector space. (The norm satisfies $\|af\| = |a|\,\|f\|$ for scalars a, $\|f+g\| \le \|f\| + \|g\|$, and $\|f\| = 0 \implies f = 0$.) For example, $L^p(X)$ for $p \in [1,\infty)$ is the completion of $C^\infty(X)$ with respect to the norm $\|f\|_{L^p} = \left[\int_X |f|^p \, d\mathrm{vol}\right]^{1/p}$. In particular, $L^1(X)$ is the space of integrable functions on X, and L^p is the space of functions whose absolute values are p-th roots of integrable functions. The space $L^\infty(X)$ is the completion of any of the $L^p(X)$ with respect to $\|f\|_{L^\infty} = \mathrm{ess}-\sup_x |f(x)|$. Another important class of examples is the space of k-times continuously differentiable functions $C^k(X)$ for k a nonnegative integer, with norm

$$\|f\|_{C^k} := \sup|f| + \sup|\nabla f| + \cdots + \sup\left|\nabla^k f\right|.$$

The derivative $\nabla^k f \in \Gamma\left((T^*X)^{\otimes k}\right)$ is the total derivative with respect to some connection on T^*X, usually taken to be the Levi-Civita connection.

A *Hilbert space* is a Banach space where the norm is induced by an inner product. The standard example is $L^2(X)$. Also we have the Sobolev spaces $H^s(X)$ which are defined for any $s \in \mathbb{R}$, but for s a non-negative integer are defined by completing $C^\infty(X)$ with respect to the norm

$$\|f\|_{H^s}^2 := \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2 + \cdots + \left\|\nabla^k f\right\|_{L^2}^2.$$

Thus $H^s(X)$ consists of functions with s derivatives in L^2.

Remark. We are not interested in the precise values of the norms in these Banach spaces. All we care about is the notion of convergence determined by a norm, i.e. its topology. Two norms $\|\bullet\|_A$ and $\|\bullet\|_B$ on a vector space V are said to be *equivalent* if there exists $C > 0$ such that for all $v \in V$,

$$C^{-1}\|v\|_A \le \|v\|_B \le C\|v\|_A.$$

This defines an equivalence relation. Equivalent norms induce the same topology. Most similarly-defined norms over a closed manifold end up being equivalent, and this gives us some flexibility in our definitions. For example, $C^k(X)$ and $H^s(X)$ do not depend on the choice of connection. Furthermore, the vast majority of our estimates will be up to some constant.

Remark. A linear map L between two Banach spaces is *continuous* if there exists some constant C such that for all f,

$$\|Lf\| \le C\|f\|.$$

Such maps are continuous in the respective topology, and thus commute with limits.

An *isometry* of Banach spaces is a linear map L such that $\|Lf\| = \|f\|$. These are not so important. What is more relevant is the notion of *isomorphism*, which is a continuous bijection satisfying the reverse of the usual inequality

$$\|f\| \leq C\,\|Lf\|\,.$$

This inequality precisely guarantees that the inverse map L^{-1} is continuous:

$$\left\|L^{-1}f\right\| \leq C\,\|f\|\,.$$

We want to understand the *elliptic estimate*, which states that if P is an elliptic differential operator of degree d, then there exists some constant $C_{P,s}$ such that for all $f \in H^{d+s}(X)$,

$$\|f\|_{H^{d+s}} \leq C_{P,s}\left(\|Pf\|_{H^s} + \|f\|_{H^s}\right).$$

Note that this estimate almost says that $P : H^{d+s}(X) \to H^s(X)$ is invertible (if not for the second term on the right). Our desired lemmas are direct consequences of the Sobolev embeddings:

1. *For $s > n/2$, $H^s(X) \subset C^0(X)$, and the embedding $H^s(X) \hookrightarrow C^0$ of Banach spaces is continuous, i.e. there exists some constant C depending on X such that for all $f \in H^s(X)$,*

   $$\|f\|_{C^0} \leq C\,\|f\|_{H^s}\,.$$

   Similarly, $H^{s+k} \hookrightarrow C^k$ is also continuous.

2. If $s > s'$, then the embedding $H^s(X) \hookrightarrow H^{s'}(X)$ is continuous and compact, i.e. if $\{f_i\}$ is a bounded sequence in $H^s(X)$, then it contains a subsequence which converges in $H^{s'}(X)$.

Corollary. *For $s > n/2$, the embedding $H^s \hookrightarrow C^0$ is compact.*

Proof. Choose s' such that $s > s' > n/2$. Then $H^s \hookrightarrow C^0$ factors as $H^s \hookrightarrow H^{s'} \hookrightarrow C^0$ which is the composition of a compact map $H^s \hookrightarrow H^{s'}$ and a continuous map $H^{s'} \hookrightarrow C^0$. It's straightforward to verify that any such composition is compact. □

Corollary. $C^\infty = \bigcap_{s\to\infty} H^s$.

Proof. Note that $C^\infty = \bigcap_{k\to\infty} C^k$. (This is not naturally a Banach space, since there is no single norm.) Since all derivatives of any smooth function are L^2, we clearly have $C^\infty \subset H^s$ for all s, so $C^\infty \subset \bigcap H^s$. Conversely,

$$C^\infty = \bigcap C^k \supset \bigcap H^{k+n/2+1} = \bigcap H^s.$$

□

Corollary (Regularity). *If P is elliptic, and $Pf \in C^\infty$, then $f \in C^\infty$. In particular, if $\Delta\omega = \alpha$ for $\omega \in L^2(X; \Lambda^p T^* X)$ and $\alpha \in \Omega^p(X)$, then $\omega \in \Omega^p(X)$.*

Proof.

$$Pf \in C^\infty \implies Pf \in H^s\ \forall s \implies \|f\|_{H^{s+d}} < \infty \implies f \in H^{s+d}\ \forall s \implies f \in C^\infty.$$

□

Corollary (Compactness). *Suppose $\{f_i\}$ is a sequence such that $\|f_i\|_{H^s}$ and $\|Pf_i\|_{H^s}$ are both bounded. Then $\{f_i\}$ has a convergent subsequence in H^s. In particular, when $s = 0$, if $\|\omega_i\|_{L^2}$ and $\|\Delta\omega_i\|_{L^2}$ are bounded, then $\{\omega_i\}$ has a convergent subsequence in $L^2(X;\Lambda^p T^* X)$.*

On a higher level, what is happening is

$$\begin{matrix}\text{Elliptic estimate+}\\ \text{Sobolev embeddings}\end{matrix} \Longrightarrow \begin{matrix}\text{Regularity+}\\ \text{Compactness}\end{matrix}$$

This same theme plays out in gauge theory as well, where modulo gauge, the equations are elliptic, and this implies that the moduli space consists of smooth fields, and either is compact or can be compactified.

Distributions on T^n

Define the n-torus $T^n := \mathbb{R}^n/(2\pi\mathbb{Z}^n)$, and take the volume measure $d\mathrm{vol} = (2\pi)^{-n}dx^1\cdots dx^n$. The inner product for complex-valued functions on T^n is $\langle f, g\rangle := \int_{T^n} f\overline{g}\, d\mathrm{vol}$, and by Fourier theory, $\{e^{ik\cdot x}\}_{k\in\mathbb{Z}^n}$ is a complete orthonormal basis, and the usual factors of 2π are absorbed by $d\mathrm{vol}$. The Fourier transform gives an isometry $L^2(T^n) \xrightarrow{\mathcal{F}} \ell^2(\mathbb{Z}^n)$ which allows us to identify $f \in L^2(T^n)$ with the sequence $\{c_k := \langle f, e^{ik\cdot x}\rangle\}_{k\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z}^n)$. Derivatives act by

$$\mathcal{F}(\partial_j f) = i\, k^j\, \mathcal{F}(f),$$

and the Laplacian is determined by

$$\mathcal{F}(\Delta f) = |k|^2\, \mathcal{F}(f).$$

We see that

$$\ker\Delta = \{c_k \mid c_k = 0 \text{ for } k \neq 0\} = \text{constants},$$

and

$$\text{image}\,\Delta = \left\{\{c_k\} \in \ell^2(\mathbb{Z}^n) \mid c_0 = 0\right\},$$

because the Green's operator is determined by

$$\mathcal{F}(Gf) = |k|^{-2}\,\mathcal{F}(f),$$

where we set $|k|^{-2} = 0$ in the exceptional case $k = 0$.

Under the Fourier transform, we get a convenient expression for

$$\begin{aligned}\|f\|_{H^s}^2 &= \|f\|_{L^2}^2 + \sum_i \|\partial_i f\|_{L^2}^2 + \sum_{i,j} \left\|\partial_i\partial_j f\right\|_{L^2}^2 + \cdots + \sum_{|\alpha|=s} \|\partial_\alpha f\|_{L^2}^2 \\ &= \sum_k \left(|c_k|^2 + \sum_i \left|k^i\right|^2 |c_k|^2 + \sum_{i,j} \left|k^i k^j\right|^2 |c_k|^2 + \cdots + \sum_{|\alpha|=s} |k^\alpha|^2 |c_k|^2\right) \\ &= \sum_k \left(1 + |k|^2 + \sum_{i,j} \left|k^i k^j\right|^2 + \cdots + \sum_{|\alpha|=s} |k^\alpha|^2\right) |c_k|^2 .\end{aligned}$$

Remember that we only care about the norm up to equivalence. The point is that the polynomial

$$1 + |k|^2 + \sum_{i,j} \left|k^i k^j\right|^2 + \cdots + \sum_{|\alpha|=s} |k^\alpha|^2$$

is positive of order $2s$, and any other weights for the coefficients $|c_k|^2$ which are positive and grow with order $2s$ will be equivalent. Thus

$$\begin{aligned} \|f\|_{H^s}^2 &\sim \sum_k \left(1 + |k|^{2s}\right) |c_k|^2 \\ &\sim \sum_k (1 + |k|)^{2s} |c_k|^2 \\ &\sim |c_0|^2 + \sum_k |k|^{2s} |c_k|^2 \\ &= \left| \int_{T^n} f \, d\text{vol} \right|^2 + \langle f, \Delta^s f \rangle . \end{aligned}$$

Note that any of the above expressions, except for the last one, make sense for any $s \in \mathbb{R}$, and we take them as the definition of $H^s(T^n)$. Specifically, by identifying a function with its Fourier coefficients,

$$\begin{aligned} H^s(T^n) &:= \left\{ \{c_k\} \mid \sum (1 + |k|)^{2s} |c_k|^2 < \infty \right\} \\ &= \left\{ \{c_k\} \mid (1 + |k|)^s c_k \in \ell^2(\mathbb{Z}^n) \right\} \\ &= (1 + |k|)^{-s} \, \ell^2(\mathbb{Z}^n). \end{aligned}$$

It's easy to verify that

$$\partial_i : H^s \to H^{s-1}$$

is continuous, $(1 + \Delta) : H^s \to H^{s-2}$ is an isomorphism, and more generally, $(\Delta - \lambda) : H^s \to H^{s-2}$ for all λ which are *not* eigenvalues of Δ. Finally, the Riesz representation theorem states that the L^2 pairing

$$\langle f, g \rangle_{L^2} = \sum_k c_k \overline{d}_k$$

is a perfect pairing, meaning that every element of $(L^2)^*$ arises as the inner product with some element of L^2. If $f \in H^s(T^n)$ with Fourier series $\{c_k\}$, and if $g \in H^{-s}(T^n)$ with Fourier series $\{d_k\}$, then $\langle f, g \rangle_{L^2}$ makes sense because

$$\text{"} \langle f, g \rangle_{L^2} \text{"} = \sum_k c_k \overline{d}_k = \sum_k (1 + |k|)^s c_k \, (1 + |k|)^{-s} \overline{d}_k \leq \|f\|_{H^s} \, \|g\|_{H^{-s}} < \infty,$$

where the $\leq$ is Cauchy-Schwarz. Moreover, since $\{(1 + |k|)^s c_k\}$ is in $\ell^2(\mathbb{Z}^n)$, any element of $(H^s)^*$ is represented as an inner product with some $\{(1 + |k|)^{-s} d_k\}$ in $\ell^2(\mathbb{Z}^n)$, i.e.

$$(H^s)^* = H^{-s}.$$

Definition. A *distribution* on T^n is an element of

$$\mathcal{D}(T^n) := \bigcup_{s \to -\infty} H^s(T^n).$$

Remark. For any $f \in \mathcal{D}(T^n)$, there is some s such that $f \in H^s(T^n)$.