We are trying to prove the Hodge decomposition

$$\Omega^p(X) = \underbrace{d\Omega^{p-1}(X) \oplus d^*\Omega^{p+1}(X)}_{\substack{\text{image } \Delta \\ (\text{but is it all of } (\ker\Delta)^\perp?)}} \oplus \underbrace{\mathcal{H}^p(X)}_{\ker\Delta}$$

into three orthogonal pieces. Note that assuming the Hodge decomposition,

$$\ker d = (\text{image } d^*)^\perp = d\Omega^{p-1}(X) \oplus \mathcal{H}^p(X),$$

so

$$\mathcal{H}^p(X) \overset{\cong}{\hookrightarrow} \ker d/\text{image } d = H^p(X;\mathbb{R}).$$

Gauge theory is essentially a nonlinear generalization of Hodge theory. When we linearize the ASD equations and the Seiberg-Witten equations, the analysis will be very similar.

At the end of last lecture, we applied Rellich's theorem to show that $\mathcal{H}^p(X)$ is finite-dimensional. It follows that it is a closed subspace (in the relative L^2 topology, or any other reasonable topology). To ensure that our decomposition is not missing any pieces, we must also show that if $\alpha \perp \ker\Delta$, then $\alpha \in \text{image }\Delta$. In other words, we must solve the PDE

$$\Delta\omega = \alpha$$

for $\omega \in \Omega^p(X)$ given $\alpha \in \mathcal{H}^p(X)^\perp \subset \Omega^p(X)$.

Lemma 1 (Rellich's Theorem)**.** *Let X be a closed oriented manifold. Suppose $\{\omega_i \in \Omega^p(X)\}_{i=1}^\infty$ is a sequence such that $\|\omega_i\|_{L^2}$ and $\|\Delta\omega_i\|_{L^2}$ are both bounded. Then ω_i contains a Cauchy subsequence, which converges in the L^2-completion of $\Omega^p(X)$, i.e.*

$$\omega_i \overset{L^2}{\to} \omega \in L^2(X;\Lambda^p(T^*X)).$$

(Passage to a subsequence is implicit.)

Lemma 2 (Regularity)**.** *Suppose $\omega \in L^2(X;\Lambda^p(T^*X))$ is a "weak solution" to "$\Delta\omega = \alpha$" in the below sense, with $\alpha \in \Omega^p(X)$. Then $\omega \in \Omega^p(X)$, i.e. ω is not just L^2 but is smooth.*

Suppose we have $\omega, \alpha \in L^2(X;\Lambda^p(T^*X))$ which satisfy the equation "$\Delta\omega = \alpha$." Since $\Delta\omega$ doesn't make sense for $\omega \in L^2$, we have to make sense of this in "weak form:"

$$\Delta\omega = \alpha \iff \forall\beta \in \Omega^p(X),\ \langle\beta, \Delta\omega\rangle = \langle\beta, \alpha\rangle \iff \forall\beta \in \Omega^p(X),\ \langle\Delta\beta, \omega\rangle = \langle\beta, \alpha\rangle.$$

This last equation now makes sense because the Laplacian hits the smooth differential form, instead of the L^2 differential form.

Philisophically, the "weak form" corresponds to viewing $\Delta\omega$ as a *distribution*, i.e. a continuous functional on the space of *smooth* forms $\Omega^p(X) \to \mathbb{R}$. Any differential p-form α with L^2 coefficients determines a unique functional ℓ_α by

$$\ell_\alpha : \Omega^p(X) \to \mathbb{R},$$
$$\ell_\alpha(\beta) := \langle\alpha, \beta\rangle.$$

For any differential operator D with smooth coefficients, we want to make sense of $D\alpha$, even when α is not differentiable. Assuming that boundary terms vanish (i.e. X is closed), then we expect "$\ell_{D\alpha}(\beta) = \langle D\alpha, \beta\rangle = \langle \alpha, D^*\beta\rangle = \ell_\alpha(D^*\beta)$" so we take as definition

$$\ell_{D\alpha}(\beta) := \ell_\alpha(D^*\beta),\ \forall \beta \in \Omega^p(X),$$

which always makes sense because β is smooth. In fact, this definition makes sense as a derivative $\ell_{D\gamma}$ for any functional $\ell_\gamma : \Omega^p(X) \to \mathbb{R}$, even when ℓ_γ does not arise from some continuous γ. Thus we are led to the dual space $\Omega^p(X)^*$ of *distributional forms*. (There is a natural topology on $\Omega^p(X)$ for which distributions are implicitly required to be continuous.) We identify L^2 forms as a subspace of $\Omega^p(X)^*$ via $\alpha \mapsto \ell_\alpha$, and we know how to extend the action of differential operators with smooth coefficients from $\Omega^p(X)$ to $\Omega^p(X)^*$.

Now suppose $\alpha \in \mathcal{H}^p(X)^\perp$. To solve our PDE $\Delta\omega = \alpha$, we define a (possibly unbounded) linear functional

$$\begin{aligned} \ell_{\Delta^{-1}\alpha} &: (\text{image } \Delta \subset \Omega^p(X)) \to \mathbb{R}, \\ &\ell_{\Delta^{-1}\alpha}(\Delta\beta) = \langle \beta, \alpha\rangle . \end{aligned}$$

We must check that this is well-defined, independent of our choice of β. But first note that as a distribution, $\ell_{\Delta^{-1}\alpha}$ formally corresponds to an inverse image under the Laplacian for α: "$\ell_{\Delta^{-1}\alpha}(\gamma) = \langle \gamma, \Delta^{-1}\alpha\rangle$," since if $\gamma = \Delta\beta$, then formally, "$\ell_{\Delta^{-1}\alpha}(\Delta\beta) = \langle \Delta\beta, \Delta^{-1}\alpha\rangle = \langle \beta, \Delta\Delta^{-1}\alpha\rangle = \langle \beta, \alpha\rangle$."

Observe that $\ell_{\Delta^{-1}\alpha}$ is well-defined, since if $\Delta\beta_1 = \Delta\beta_2$, then $\Delta(\beta_1 - \beta_2) = 0$ so $\beta_1 - \beta_2 \in \mathcal{H}^p(X)$, thus $\langle \beta_1, \alpha\rangle = \langle \beta_2, \alpha\rangle$ since $\alpha \in \mathcal{H}^p(X)^\perp$.

Lemma 3. *The functional $\ell_{\Delta^{-1}\alpha} : (\text{image } \Delta) \to \mathbb{R}$, is L^2-bounded, i.e.*

$$|\ell_{\Delta^{-1}\alpha}(\gamma)| \leq C_\alpha \|\gamma\|_{L^2}$$

for some C_α which is independent of γ.

Assuming this lemma, then by the Hahn-Banach theorem, $\ell_{\Delta^{-1}\alpha}$ extends to a bounded linear functional $\tilde{\ell}_{\Delta^{-1}\alpha} : L^2(X; \Lambda^p T^*X) \to \mathbb{R}$. By the Riesz representation theorem, there is some $\omega \in L^2(X; \Lambda^p T^*X)$ such that $\tilde{\ell}_{\Delta^{-1}\alpha}(\gamma) = \langle \omega, \gamma\rangle$. This ω then satisfies "$\Delta\omega = \alpha$" weakly, since

$$\langle \Delta\omega, \gamma\rangle := \langle \omega, \Delta\gamma\rangle = \tilde{\ell}_{\Delta^{-1}\alpha}(\Delta\gamma) = \ell_{\Delta^{-1}\alpha}(\Delta\gamma) = \langle \alpha, \gamma\rangle .$$

Since we assumed $\alpha \in \Omega^p(X)$, by Lemma 2 we conclude that $\omega \in \Omega^p(X)$. Thus $\Delta\omega = \alpha$, and we are done.

To prove Lemma 3, we will need

Lemma 4 (Poincaré inequality)**.** *Let X be a closed oriented manifold. There is a constant C_X such that*

$$\|\beta\|_{L^2} \leq C_X \|\Delta\beta\|_{L^2}\ \ \forall \beta \in \mathcal{H}^p(X)^\perp.$$

Proof of Lemma 4. By contradiction, supppose not. Then there is some sequence $\beta_i \in \mathcal{H}^p(X)^\perp$ with $\|\beta_i\|_{L^2} = 1$ and $\|\Delta\beta_i\|_{L^2} \to 0$. By Lemma 1, passing to a subsequence, $\beta_i \to \beta$ for some $\beta \in L^2(X; \Lambda^p T^*X) \cap \mathcal{H}^p(X)^\perp$. Furthermore, "$\Delta\beta = 0$" since for any fixed $\alpha \in \Omega^p(X)$, "$\langle \Delta\beta, \alpha\rangle$"$=$ $\langle \beta, \Delta\alpha\rangle = 0$, since

$$|\langle \beta, \Delta\alpha\rangle| = \left|\left\langle \lim_i \beta_i, \Delta\alpha \right\rangle\right| = \lim_i |\langle \Delta\beta_i, \alpha\rangle| \le \|\alpha\|_{L^2} \lim_i \|\Delta\beta_i\|_{L^2} \to 0.$$

By Lemma 2, since $0 \in \Omega^p(X)$ we have $\beta \in \Omega^p(X) \subset L^2(X; \Lambda^p T^*X) \cap \mathcal{H}^p(X)^\perp$ and satisfies $\Delta\beta = 0$, so $\beta \in \mathcal{H}^p(X)^\perp \cap \mathcal{H}^p(X)$, thus $\beta = 0$. But $\|\beta\|_{L^2} = \lim \|\beta_i\|_{L^2} = 1$, which is a contradiction. □

Proof of Lemma 3. By contradiction, suppose not. Then there is some sequence $\{\gamma_i \in \text{image } \Delta\}$ with $\|\gamma_i\|_{L^2} = 1$ but $\ell_{\Delta^{-1}\alpha}(\gamma_i) \to \infty$. Choose $\gamma_i = \Delta\beta_i$. Since $\mathcal{H}^p(X)$ is finite-dimensional, we can assume $\beta_i \in \mathcal{H}^p(X)^\perp$ after subtracting the appropriate finite linear combination. Then $\langle \beta_i, \alpha\rangle = \ell_{\Delta^{-1}\alpha}(\Delta\beta_i) = \ell_{\Delta^{-1}\alpha}(\gamma_i) \to \infty$ while $\|\Delta\beta_i\|_{L^2} = 1$. By Lemma 4, $\|\beta_i\|_{L^2} \le C_X$. Thus by Lemma 1, after passing to a subsequence, $\beta_i \to \beta \in L^2(X; \Lambda^p T^*X)$, so $\langle \beta_i, \alpha\rangle \to \langle \beta, \alpha\rangle =$ finite, which is a contradiction. □

Up to the proofs of Lemma 1 and Lemma 2 we have proved the Hodge decomposition.

At this point, we can define the *Green's operator* $G : \Omega^p(X) \to (\mathcal{H}^p(X))^\perp \subset \Omega^p(X))$ by defining $G(\alpha) := \omega$ to be the unique solution to $\Delta\omega = \alpha - \pi_{\mathcal{H}^p}(\alpha)$. It's a simple exercise to show that G is a bounded linear self-adjoint operator which commutes with d, d^*, and Δ. Since G is bounded, it extends to the L^2 completion

$$G : L^2(X; \Lambda^p T^*X) \to (\mathcal{H}^p(X))^\perp \subset L^2(X; \Lambda^p T^*X)),$$

and satisfies

$$\begin{aligned} \Delta G &= \mathrm{Id}_{L^2(X;\Lambda^p T^*X)} - \pi_{\mathcal{H}^p}, \\ G\Delta &= \mathrm{Id}_{\Omega^p} - \pi_{\mathcal{H}^p}. \end{aligned}$$

Definition. A linear operator between Banach or Hilbert spaces is *compact* if the image of any bounded sequence has a convergent subsequence.

A simple consequence of the Rellich lemma is that the Green's operator G is compact. The spectral theorem for compact self-adjoint operators then yields an eigenspace decomposition of $L^2(X; \Lambda^p T^*X)$ into finite-dimensional eigenspaces of G with eigenvalues μ_i which accumulate only to zero. Since Δ is inverse to G on the complement of $\mathcal{H}^p(X)$, Δ has the same eigenspace decomposition, with eigenvalues $\lambda_i = 1/\mu_i$. Moreover, the eigenvalues λ_i of Δ can accumulate only towards infinity.

Elliptic operators

The prototypical elliptic operator over a closed manifold is the Laplacian $\Delta = -\sum_{i=1}^n \partial_i^2$ over the n-torus $\mathbb{R}^n/(2\pi i\mathbb{Z}^n)$, acting on smooth functions $\Delta : \Omega^0(\mathbb{T}^n) \to \Omega^0(\mathbb{T}^n)$. This case is worth understanding completely, because it very concretely exhibits the properties of elliptic operators. More generally we have the Hodge Laplacian, Dirac operator, and more general elliptic operators.

The main theorems are

Theorem. *If X is a closed manifold, and P is an elliptic differential operator, then* $\ker P$ *is finite-dimensional, P^* is also elliptic, and image $P = (\ker P^*)^\perp$.*

Theorem. *If X is a closed manifold, and P is an elliptic differential operator of degree d, then for each $s \in \mathbb{R}$, there exists a constant $C_{P,s}$ such that*

$$\|\omega\|_{H^{s+d}} \le C_{P,s} \|P\omega\|_{H^s} \quad \forall \omega \in H^{s+d},\ \omega \perp \ker P,$$
$$\|\omega\|_{H^{s+d}} \le C_{P,s} \left(\|P\omega\|_{H^s} + \|\omega\|_{H^s}\right) \quad \forall \omega \in H^{s+d},$$

where the H^s denotes the Sobolev spaces of distributions with s derivatives in L^2.