Classification of vector bundles over a 4-manifold

Recall from last time that if G is any compact simple Lie group, then G is the quotient of a simply-connected simple Lie group $\tilde{G}$, which is uniquely determined by the Dynkin diagram of $\mathfrak{g}$. The quotient is determined by $\ell := \ker(\exp : \mathfrak{g} \to G)$, which can be any intermediate lattice

$$\Lambda_{\text{root}} \subset \ell \subset \Lambda_{\text{weight}}.$$

and so $\pi_1(G) = \ell/\Lambda_{\text{root}}$ is a finite abelian group.

Generalized Stiefel-Whitney class

Definition. For any smooth principal G-bundle, we define the *generalized Stiefel-Whitney class* $w_2(P) \in H^2(X; \pi_1(G))$ by the following procedure. The short exact sequence

$$0 \to \pi_1(G) \to \tilde{G} \to G \to 0$$

induces the map

$$\check{H}^1(X; G_{C^\infty}) \stackrel{w_2}{\to} H^2(X; \pi_1(G)),$$

which obstructs extending the structure group of P to $\tilde{G}$.

Note that if $G = \mathrm{SO}(k)$ for $k \geq 3$, then $\tilde{G} = \mathrm{Spin}(k)$ and $\pi_1(G) = \mathbb{Z}_2$. Then $w_2(P) \in H^2(X; \mathbb{Z}_2)$ is the ordinary second Stiefel-Whitney class, which is the obstruction to finding a $\mathrm{Spin}(k)$ structure.

Instanton number

We review the definition of connected sum of two connected oriented n-manifolds $X_1 \# X_2$. Choose two small smooth balls $B_i \subset X_i$, with B_1 oriented positively and B_2 oriented negatively. Delete the centers from the B_i, and in polar coordinates, identify $r\phi \in B_1$ with $(1-r)\phi \in B_2$. Flipping the radial direction reverses orientation, which is compensated by the negative orientation on B_2. The result is unique up to diffeomorphism and independent of choices.

The operation of connected sum can be extended from oriented manifolds to oriented manifolds equipped with principal G-bundles. $P_i \to X_i$. We simply choose trivializations of P_i over B_i, and glue according to the trivializations. The isomorphism class of $P_1 \# P_2$ is independent of choice of trivialization. Since S^n is the identity with respect to connect sum, if we connect sum with a principal bundle over S^4 we can alter the principal bundle without changing the base.

Let G be a compact simple Lie group. Consider a principal G-bundle P_k over S^4. Since the thickened hemispheres H_+ and H_- are contractible, we can find local trivializations of P over H_+ and H_-. The isomorphism class of P is determined by the single transition function $\phi : H_+ \cap H_- \to G$. The isomorphism class of P depends only on the homotopy class of ϕ. The intersection $H_+ \cap H_-$ is homotopy equivalent to the equatorial S^3, so we get a correspondence between

$$\left\{ \substack{\text{isomorphism classes of smooth} \\ \text{principal } G\text{-bundles over } S^4} \right\} \longleftrightarrow \left\{ \text{homotopy classes of maps } S^3 \to G \right\} =: \pi_3(G).$$

Bott showed that when G is simple, $\pi_3(G) \cong \mathbb{Z}$. If we let k index the corresponding integer, then we get a group structure $P_{k_1} \# P_{k_2} \cong P_{k_1+k_2}$. The number k is called the *instanton number, Pontryagin index,* or *topological charge*. It's possible to compute $k(P)$ by a curvature integral, or by characteristic classes. One such formula is

$$k(P) = -\frac{\langle p_1(P \times_{\text{ad}} \mathfrak{g}), [X] \rangle}{2\check{h}(\mathfrak{g})},$$

where $p_1(\text{real vector bundle}) \in H^4(X;\mathbb{Z})$ is the Pontryagin class, and $\check{h}(\mathfrak{g})$ is the "dual Coxeter number." These formulas make sense over general closed oriented X. However because of the denominator, k is not always an integer. It satisfies two important properties.

1. If $P_{k'}$ denotes the bundle of charge k' over S^4, then

$$k(P \# P_{k'}) = k(P) + k'.$$

2. The value of $k(P) \pmod 1$ is determined by $w_2(P)$. In particular,

$$k(P) \equiv -\tfrac{1}{2} w_2(P)^2 \pmod 1.$$

   However this formula requires caution, since $w_2(P)$ has coefficients in $\pi_1(G)$. Converting $w_2(P)^2$ to a number implicitly involves a choice of metric on Λ_{weight}, which in generally introduces further denominators.

Theorem (Dold-Whitney). *If X is a compact oriented 4-manifold, and if G is a compact simple Lie group, then isomorphism classes of principal G-bundles $P \to X$ are in bijection with pairs $(k, w_2) \in \mathbb{Q} \times H^2(X;\pi_1(G))$ satisfying the condition*

$$k \equiv -\tfrac{1}{2} {w_2}^2 \pmod 1.$$

The most important cases are $G = \mathrm{SU}(2)$, $G = \mathrm{SO}(3)$, and $G = \mathrm{U}(2)$ (not simple).

- In the case $G = \mathrm{SU}(r)$, $w_2 = 0$ and $k \in \mathbb{Z}$ with $k(P) = c_2(P) \cdot [X]$, where $c_2(P) \in H^4(X;\mathbb{Z})$ is the second Chern class.

- In the case $G = \mathrm{SO}(3)$, $k \in \frac{1}{4}\mathbb{Z}$ with $k(P) = -\frac{1}{4} p_1(P) \cdot [X] \equiv -\frac{1}{4} {w_2}^2(P) \pmod 1$, where $p_1(P) \in H^4(X;\mathbb{Z})$ is the first Pontryagin class, and $w_2(P) \in H^2(X;\mathbb{Z}_2)$, and the extra factor of $\frac{1}{2}$ in front of ${w_2}^2(P)$ arises from the identification $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$. Note that we would only expect ${w_2}^2(P) \in H^4(X;\mathbb{Z}_2)$ to be well-defined modulo 2, but in fact it is well-defined modulo 4 by computing ${\tilde{w}_2}^2(P)$ for some $\tilde{w}_2 \in H^2(X;\mathbb{Z})$ which reduces modulo 2 to $w_2 \in H^2(X;\mathbb{Z}_2)$.

- In the case $G = \mathrm{U}(2)$, $k \in \frac{1}{2}\mathbb{Z}$ with $k(P) = (c_2(P) - \frac{1}{2} c_1(P)^2) \cdot [X]$.

Note that for any compact Lie group G, the universal covering $\tilde{G} \cong \mathbb{R}^k \times G_1 \times \cdots\cdots G_k$, where the G_i are compact simple simply connected Lie groups, so although it's beyond the scope of this course, classification of the corresponding principal bundles is still accessible.

Hodge star

Let X be a closed oriented Riemannian n-manifold X. Consider the de Rham cohomology $H^k(X;\mathbb{R})$ defined by

$$\frac{\ker d \subset \Omega^k(X)}{\text{image } d}.$$

We wish to find a natural subspace $\mathcal{H}^k \subset \Omega^k(X)$ such that $\mathcal{H}^k \cong H^k(X;\mathbb{R})$ via the map $\omega \mapsto [\omega]$. In other words, we want to trade our quotient space $H^k(X;\mathbb{R})$ for a subspace $\mathcal{H}^k$.

If V is a finite-dimensional Euclidean vector space, and if $W \subset V$ is a subspace, then we can naturally represent the quotient V/W by $W^\perp$. Specifically, each coset in V/W intersects a unique vector in $W^\perp$, so we get an isomorphism $W^\perp \to V/W$ by $v \mapsto [v]$. Of course $W^\perp$ is not the only subspace with this property.

Definition. A subspace $S \subset V$ is called a *slice* for the quotient V/W if the quotient map restricts to S as an isomorphism.

The idea of the Hodge decomposition is simply to imitate this construction in the infinite dimensional setting of de Rham theory.

The first ingredient we need is an inner product on $\Omega^p(X)$. For this, consider $\mathbb{R}^n$ equipped with the standard $\mathrm{SO}(n)$ structure, i.e. the standard Euclidean metric and orientation, so that $\{e^1, \dots, e^n\}$ is an orthonormal basis. We define a Euclidean metric on $\Lambda^p\mathbb{R}^n$ by declaring $e^{i_1} \wedge \cdots \wedge e^{i_p}$ to be an orthonormal basis. More invariantly, one can define

$$\left\langle v^1 \wedge \cdots \wedge v^p, w^1 \wedge \cdots \wedge w^p \right\rangle := \det \left\langle v^i, w^j \right\rangle,$$

and the right hand side is clearly invariant under $\mathrm{O}(n)$. (The action is $v_1 \wedge \cdots \wedge v_p \mapsto gv_1 \wedge \cdots \wedge gv_p$.)

We can define a map on the exterior powers of $\mathbb{R}^n$ by $\star : \Lambda^p\mathbb{R}^n \to \Lambda^{n-p}\mathbb{R}^n$ characterized by the relation

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \, e^1 \wedge \cdots \wedge e^n.$$

This characterization is clearly invariant under $\mathrm{SO}(n)$, and one computes that

$$\star \left(e^{i_1} \wedge \cdots \wedge e^{i_k} \right) = \pm e^{\tilde{i}_1} \wedge \cdots \wedge e^{\tilde{i}_{n-p}},$$

where $\tilde{i}$ denotes the indices complementary to i, and $\pm$ is determined by

$$e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge e^{\tilde{i}_1} \wedge \cdots \wedge e^{\tilde{i}_{n-p}} = \pm e^1 \wedge \cdots \wedge e^n.$$

One verifies that

$$\star^2 = (-1)^{p(n-p)} : \Lambda^p\mathbb{R}^n \to \Lambda^p\mathbb{R}^n,$$

Furthermore, $\star$ encodes the orientation and metric via the identities

$$\star 1 = e^1 \wedge \cdots \wedge e^n,$$
$$\langle \alpha, \beta \rangle = \star(\alpha \wedge \star\beta) = \star(\beta \wedge \star\alpha).$$

The Hodge star map is equivariant under $\mathrm{SO}(n)$, i.e. $\star(g\alpha) = g(\star\alpha)$. For any vector space V equipped with a reduction to $\mathrm{SO}(n)$, i.e. V is equipped with an orientation and a Euclidean metric, the Hodge star determines a map $\star : \Lambda^p V \to \Lambda^{n-p} V$, where

$$\Lambda^p V := \mathrm{Fr}^{\mathrm{SO}}(V) \times_\rho \Lambda^p \mathbb{R}^n.$$

Of course this also makes sense for any principal $\mathrm{SO}(n)$ bundle. Suppose X is a smooth n-manifold equipped with a reduction of the cotangent bundle T^*X to a $\mathrm{SO}(n)$ structure, i.e. X is oriented Riemannian. (A Riemannian metric determines an isomorphism $TX \to T^*X$, so reductions of T^*X or TX are equivalent.) In particular, $\star$ induces a bundle map $\Lambda^p T^*X \to \Lambda^{n-p} T^*X$. Differential forms are sections $\Omega^p(X) = \Gamma(\Lambda^p T^*X)$, so we get a map $\star : \Omega^p(X) \to \Omega^{n-p}(X)$ which acts fiberwise.

Finally, we define a Euclidean inner product on $\Omega^p_c(X)$ (p-forms with compact supports) by

$$\langle \alpha, \beta \rangle := \int_X \alpha \wedge \star\beta.$$