We begin with a brief overview of the main ideas, mainly intended to expose beginners to new language. This will be followed with more context and background, which should be more accessible to non-experts.

By default, all manifolds are assumed to be closed (i.e. compact and without boundary) and oriented.

Donaldson Theory

In four dimensions, there are infinite families of smooth manifolds which are homeomorphic, but not diffeomorphic.

$$X_i \overset{\text{diffeo}}{\nsim} X_j \quad X_i \overset{\text{homeo}}{\sim} X_j$$

While the topology is the same, the notion of *calculus* is different. Thus to distinguish such manifolds, it makes sense to study solutions to differential equations. Gauge theory provides specific differential equations which are sufficiently natural and simple for this purpose.

The most basic differential equation from gauge theory is the *Yang-Mills equation*

$$d_A \star F_A = 0,$$

which is a second-order PDE for a connection A. Here A is a connection on a principle bundle P. In the simplest case where the gauge group of P is $\mathrm{U}(1)$, A is physically the electromagnetic potential, and $F_A := dA$ is the electromagnetic field. We will be interested in the next simplest case, where the gauge group is $\mathrm{SU}(2)$, and A is the gauge field corresponding to the weak force. The field is $F_A := dA + \frac{1}{2}[A \wedge A]$.

In either case, the Yang-Mills equation arises as the Euler-Lagrange equation for stationary points of the Yang-Mills energy

$$L_{\mathrm{YM}} = \int_X |F_A|^2 \, d\mathrm{vol}.$$

In physics it is common to decompose the electromagnetic field into orthonormal electric and magnetic parts, which depend on a space-time basis $|F_A|^2 = |E|^2 + |B|^2$. In four dimensions with Euclidean signature, there is an orthonormal splitting $F_A = F_A^+ + F_A^-$ which is basis-independent. This leads to the a*nti-self-dual Yang-Mills equation* or *instanton equation*

$$F_A^+ = 0,$$

which is first-order in A.

The decomposition satisfies

$$\int_X \left(|F_A^-|^2 - \int_X |F_A^+|^2 \right) d\mathrm{vol} = C_{\mathrm{top}},$$

where C_{top} is a constant depending on the topology of the principal bundle. Consequently,

$$L_{\mathrm{YM}} = \int_X \left(|F_A^+|^2 + |F_A^-|^2 \right) = C_{\mathrm{top}} + \int_X 2\,|F_A^+|^2 .$$

Connections satisfying this equation are absolute minimizers of L_{YM}, and thus also satisfy the second-order Yang-Mills equation.

For any fixed bundle P, the space of connections $\mathcal{A}_P$ is an affine linear space, which is rather boring. However, the gauge group

$$\mathcal{G}_P := \mathrm{Aut}(P)$$

acts on $\mathcal{A}_P$, and the quotient space of physical states

$$\mathcal{B}_P := \mathcal{A}_P/\mathcal{G}_P$$

has interesting topology.

Informally speaking, the equation $F_A^+ = 0$ cuts out a finite-dimensional oriented submanifold

$$\mathcal{M}_{\mathrm{ASD}} \subset \mathcal{B}_P.$$

Thus it defines a homology class

$$[\mathcal{M}_{\mathrm{ASD}}] \subset H_d(\mathcal{B}_P),$$

where

$$d = \dim \mathcal{M}_{\mathrm{ASD}}.$$

(Disclaimer: in reality, the situation is much more complicated due to technical issues, but I'm describing the general idea.) We can produce numbers by evaluating this on cohomology classes in $H^d(\mathcal{B}_P)$. This gives us the Donaldson invariants, which often detect smooth structures.

Note: The equations, and thus the resulting moduli space, depend on a choice of Riemannian metric on X. The Donaldson invariants are constructed in such a way that they are (usually) independent of this choice. Instead, they depend only on the underlying differential-topological structure on X.

Warning: These invariants are often called *topological* invariants. However, they are *not* well-defined in the sense of point-set topology! Instead, they are *differential-topological* invariants.

Accomplishments of this theory include:

- Large classes of topological four-manifolds which admit no smooth structure
- Examples of "exotic" topological four-manifolds which admit multiple smooth structures, including $\mathbb{R}^4$
- Invariants which are sometimes capable of distinguishing smooth structures

Kevin Iga [1] nicely summarizes the development of gauge theory as a tool in four-dimensional differential topology:

1983: Donaldson's Theorem on intersection form, and Donaldson invariants
1988: Witten's connection with $\mathcal{N} = 2$ SUSY [Beyond scope of this course]
1994: Seiberg-Witten theory

> In 1983, Donaldson shocked the topology world by using instantons from physics to prove new theorems about four-dimensional manifolds, and he developed new topological invariants. In 1988, Witten showed how these invariants could be obtained by correlation functions for a twisted $\mathcal{N} = 2$ SUSY gauge theory. In 1994, Seiberg and Witten discovered dualities for such theories, and in particular, developed a new way of looking at four-dimensional manifolds that turns out to be easier, and is conjectured to be equivalent to, Donaldson theory.

Seiberg-Witten invariants follow the same scheme but with different equations:

$$F_A^+ = q(\phi),$$
$$\partial\!\!\!/_A \phi = 0,$$

where ϕ is a spinor, q is a quadratic map (unique up to a constant), and A is a $\mathrm{U}(1)$ connection (or electromagnetic potential). While this equation is conceptually more complicated due to the coupling with the spinor, the technicalities are vastly simpler. Seiberg-Witten invariants are conjectured to encode the same information as Donaldson invariants, and this conjecture has been rigorously established in most cases.

Since Donaldson theory was mostly replaced by Seiberg-Witten theory, one might question why I should present it. Not only is it historically interesting, but the past few years have seen a resurgence in Donaldson theory.

By assuming a few identities, right now I can prove some powerful theorems using Seiberg-Witten theory.

Define the Seiberg-Witten energy

$$L_{\mathrm{SW}}(A, \phi) := \int_X \left(\left|\partial\!\!\!/_A \phi\right|^2 + \left|F_A^+ - q(\phi)\right|^2 \right) d\mathrm{vol}.$$

Then $L_{\mathrm{SW}}(A, \phi) \geq 0$, with equality exactly when the Seiberg-Witten equations are satisfied. Expanding,

$$L_{\mathrm{SW}}(A, \phi) = \int_X \left(\left|\partial\!\!\!/_A \phi\right|^2 + \left|F_A^+\right|^2 + |q(\phi)|^2 - 2\left\langle F_A^+, q(\phi)\right\rangle \right) d\mathrm{vol}.$$

There's an identity from differential geometry called the Lichnerowicz–Weitzenböck formula, which states that

$$0 = \int_X \left(|\nabla_A \phi|^2 - \left|\partial\!\!\!/_A \phi\right|^2 + \tfrac{1}{4} s\, |\phi|^2 + 2\left\langle F_A^+, q(\phi)\right\rangle \right) d\mathrm{vol},$$

where the function s is scalar curvature. Taking this as given, and adding it to $L_{\mathrm{SW}}(A, \phi)$, we get

$$L_{\mathrm{SW}}(A, \phi) = \int_X \left(|\nabla_A \phi|^2 + \tfrac{1}{4} s\, |\phi|^2 + \left|F_A^+\right|^2 + |q(\phi)|^2 \right) d\mathrm{vol}.$$

Another useful identity is $|q(\phi)|^2 = \frac{1}{8} |\phi|^4$.

$$L_{\mathrm{SW}}(A, \phi) = \int_X \left(|\nabla_A \phi|^2 + \left|F_A^+\right|^2 + \tfrac{1}{4} s\, |\phi|^2 + |\phi|^4 \right) d\mathrm{vol}.$$

Note that if $s \geq 0$ everywhere, then the only way this integral can be zero is if $\phi \equiv 0$. In most generic cases, it is impossible to have solutions with $\phi \equiv 0$. Thus, if $s \geq 0$, then all the Seiberg-Witten invariants vanish.

Recall that $\int_X \left|F_A^+\right|^2 = -\frac{1}{2} C_{\mathrm{top}} + \frac{1}{2} \int_X |F_A|^2$. Thus

$$L_{\mathrm{SW}}(A, \phi) = -\tfrac{1}{2} C_{\mathrm{top}} + \int_X \left(|\nabla_A \phi|^2 + |F_A|^2 + \tfrac{1}{4} s\, |\phi|^2 + \tfrac{1}{8} |\phi|^4 \right) d\mathrm{vol}.$$

Completing the square,

$$L_{\mathrm{SW}}(A, \phi) = \int_X \left(|\nabla_A \phi|^2 + |F_A|^2 + \tfrac{1}{8} \left(|\phi|^2 - (-s) \right)^2 \right) d\mathrm{vol} - \left(\tfrac{1}{2} C_{\mathrm{top}} + \tfrac{1}{8} \int_X s^2 \right).$$

This is an extremely powerful form of the energy. In particular, we have the identity

$$\left(\tfrac{1}{2} C_{\mathrm{top}} + \tfrac{1}{8} \int_X s^2 \right) \leq \int_X \left(|\nabla_A \phi|^2 + |F_A|^2 + \tfrac{1}{8} \left(|\phi|^2 - (-s) \right)^2 \right),$$

with equality iff (A, ϕ) is a solution. Fixing the topology and geometry, the left-hand side becomes a constant, independent of (A, ϕ). The right hand side is the sum of three positive terms. Thus for any solution, we have the following bounds:

$$\int_X |\nabla_A \phi|^2, \int_X |F_A|^2, \int_X \tfrac{1}{8}\left(|\phi|^2 - (-s)\right)^2 \leq C,$$

where

$$C := \left(\tfrac{1}{2} C_{\text{top}} + \tfrac{1}{8}\int_X s^2\right).$$

We can find a bound on $\int |\phi|^4$ as follows. For dealing with cross-terms, it's useful to have the inequality

$$|xy| = \tfrac{1}{2}\left(\varepsilon x^2 + \varepsilon^{-1} y^2 - \varepsilon^{-1}(\varepsilon|x| - |y|)^2\right) \leq \tfrac{1}{2}\left(\varepsilon x^2 + \varepsilon^{-1} y^2\right).$$

Undoing the completion of the square,

$$\int_X \left(|\phi|^4 + 2s|\phi|^2\right) \leq 4C_{\text{top}}.$$

Thus

$$\int_X \left((1-\varepsilon)|\phi|^4 - \varepsilon^{-1}s^2\right) \leq 4C_{\text{top}},$$

$$\int_X |\phi|^4 \leq \frac{4C_{\text{top}} + \varepsilon^{-1}\int_X s^2}{1-\varepsilon}.$$

Setting $\varepsilon = \frac{1}{2}$ yields

$$\int_X |\phi|^4 \leq 8C_{\text{top}} + 4\int_X s^2.$$

It's possible to choose ε more cleverly, but it's rarely worth the effort.

After some simple Sobolev theory, this readily implies compactness of the moduli space.

The *minimal genus problem* asks: given an integral homology class of degree two inside a smooth four-manifold, what is the minimal genus of a smoothly embedded Riemann surface Σ which represents it? Some simpler homology classes might be representable by smoothly embedded spheres or tori, while more complicated classes may require surfaces of higher genus.

In the case when our four-manifold is a complex manifold and Σ is a complex submanifold, the genus $g(\Sigma)$ is determined by the *adjunction formula*

$$2g(\Sigma) - 2 = \langle K, [\Sigma]\rangle + [\Sigma]\cdot[\Sigma].$$

For example on $\mathbb{CP}^2$, $K = -3[H]$, $[\Sigma] = d[H]$, and $[H]\cdot[H] = 1$, where d is the degree of Σ. This yields the classical formula $g(\Sigma) = \frac{1}{2}(d-1)(d-2)$. The *Thom conjecture* asserts that inside $\mathbb{CP}^2$, the genus does not decrease if we allow smooth (but non-algebraic) Σ. Kronheimer and Mrowka solved the Thom conjecture by discovering a simple Seiberg-Witten proof. Their idea yields a beautiful extension of the adjunction formula to an *adjunction inequality* for non-algebraic manifolds, where the Seiberg-Witten invariants produce lower bounds for $g(\Sigma)$.

Seiberg-Witten invariants depend on a choice of something called a Spin^c-structure $\mathfrak{s}$, and $\mathrm{SW}(\mathfrak{s}) \in \mathbb{Z}$. For each $\mathfrak{s}$ there is an associated Chern class $c_1(\mathfrak{s}) \in H^2(X;\mathbb{Z})$.

Suppose X is a smooth oriented closed 4-manifold with $b_2^+ > 1$. Suppose Σ is an embedded surface of genus g with $\Sigma\cdot\Sigma \geq 0$. Suppose κ is a Seiberg-Witten basic class for X. If Σ is not torsion and $g \geq 1$, then

$$2g - 2 \geq |\kappa\cdot\Sigma| + \Sigma\cdot\Sigma$$

.

The idea of the proof is quite nice. Suppose that $[\Sigma] \cdot [\Sigma] = 0$. (If $[\Sigma] \cdot [\Sigma] > 0$ then we can reduce to this case by a blowup argument.) This means that the normal bundle of Σ is trivial, so that Σ contains a tubular neighborhood of the form $\Sigma \times D^2 \hookrightarrow X$. Inside of D^2 we can find an annulus $S^1 \times [0, 1]$. Inside this annulus is a smaller disk $\tilde{D}^2$. We can decompose X into three pieces: the annulus, and the two complementary pieces on either side:

$$X = X' \cup \left(\Sigma \times S^1 \times [0, 1]\right) \cup \left(\Sigma \times \tilde{D}^2\right),$$

where $X' := X \backslash (\Sigma \times D^2)$. Under the assumption that the Seiberg-Witten invariant is nonzero, we are guaranteed at least one solution to the Seiberg-Witten equations for *any* choice of Riemannian metric. We fix a nice metric on $\Sigma \times S^1$, and use the product metric on the "neck" $\Sigma \times S^1 \times [0, L]$, parameterized by the length L. We extend to fixed metrics on X' and $\left(\Sigma \times \tilde{D}^2\right)$. By sending $L \to \infty$, the behavior of the solution along the neck becomes dominant.

By the Gauss-Bonnet theorem, Riemannian scalar curvature along the neck is proportional to the Euler characteristic $\chi(\Sigma)$:

$$\|\mathrm{scal}\|^2_{L^2} \sim L\,\chi(\Sigma)^2.$$

Using identities of Riemannian geometry, the Lagrangian for the Seiberg-Witten equations can be rewritten so that the spinor is coupled with the Riemannian scalar curvature. This leads to the inequality

$$\|F_A^+\|_{L^2} \overset{\mathrm{SW}}{=} \|(\phi \otimes \phi^*)_0\|_{L^2} \leq 8^{-1/2}\,\|\mathrm{scal}\|_{L^2}\,.$$

Finally,

$$\|F_A^+\|^2_{L^2} \sim \tfrac{1}{2}\,\|F_A\|^2_{L^2} \sim L \cdot C^2,$$

where C is proportional to a topological number $|\langle c_1(\mathfrak{s}), [\Sigma]\rangle|$.

In conclusion,

$$-\sqrt{L}\chi(\Sigma) \sim \|\mathrm{scal}\|_{L^2} \gtrsim \|F_A^+\|^2_{L^2} \sim \sqrt{L}C.$$

More precisely, we get an inequality of the type

$$-\sqrt{L}\chi(\Sigma) \geq \sqrt{L}\,|\langle c_1(\mathfrak{s}), [\Sigma]\rangle| + \text{some constant independent of } L.$$

Sending $L \to \infty$ and being more careful with constants of proportionality, we obtain

$$-\chi(\Sigma) = 2\,g(\Sigma) - 2 \geq |\langle c_1(\mathfrak{s}), [\Sigma]\rangle|\,.$$

In the more general case $[\Sigma] \cdot [\Sigma] \geq 0$, the blowup argument produces

$$2\,g(\Sigma) - 2 \geq |\langle c_1(\mathfrak{s}), [\Sigma]\rangle| + [\Sigma] \cdot [\Sigma]\,,$$

which is the celebrated adjunction inequality. It remains to explain the objects involved in this proof, and also to establish the analytic estimates which make this argument rigorous.

The plan for the remainder of this course is to fill in most details of everything discussed so far.

Intro to cohomology

The simplest differential invariants of smooth manifolds come from de Rham cohomology. Before discussing classification of manifolds, it makes sense to understand cohomology so that we have a useful tool.

In some sense, cohomology is the study of locally constant functions. Real-valued locally constant functions on a manifold form a vector space of dimension equal to the number of connected components. This may seem like the end of the story, but surprisingly it's not. For further analysis, we will need some linear/homological algebra.

Consider a vector subspace $W \subset V$. There are two primary ways to describe W. We can view it either by a parameterization:

$$W = \text{image}(V' \to V).$$

or by defining equations:

$$W = \ker(V \to V'').$$

It is useful to switch between these viewpoints. For instance, the line in $\mathbb{R}^3$ given by

$$\text{span/image}\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} : \mathbb{R}^1 \to \mathbb{R}^3$$

is equivalent to

$$\text{nullspace}/\ker\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 3 & 2 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^3.$$

Each row represents the intersection of a plane. In general we would expect three planes to intersect in a point, but the rows are linearly dependent. The condition that the line is cut out by the hyperplanes is that

$$\ker\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 3 & 2 \end{pmatrix} = \text{image}\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}.$$

The inclusion image $\subset$ ker is the condition that the composition of the two maps gives zero:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The opposite inclusion ker $\subset$ image is the statement that

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 3 & 2 \end{pmatrix}\vec{v} = 0 \implies \vec{v} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}\tilde{\vec{v}}.$$

Next we should describe the defect in the 3×3 matrix. The relation among the rows is given by the matrix

$$\begin{pmatrix} -2 & 1 & 1 \end{pmatrix}.$$

Because there is no further defect, we stop. What we now have is a sequence of linear maps

$$\cdots \to \mathbb{R}^0 \longrightarrow \mathbb{R}^1 \xrightarrow{\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 3 & 2 \end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix} -2 & 1 & 1 \end{pmatrix}} \mathbb{R}^1 \longrightarrow \mathbb{R}^0 \longrightarrow \mathbb{R}^0 \to \cdots$$

where the kernel of any linear map corresponds to the image of the previous. This is called an *exact sequence*. The idea is that either side

$$\cdots \to \mathbb{R}^0 \longrightarrow \mathbb{R}^1 \xrightarrow{\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}} \mathbb{R}^3$$

or

$$\mathbb{R}^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 3 & 2 \end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix} -2 & 1 & 1 \end{pmatrix}} \mathbb{R}^1 \longrightarrow \mathbb{R}^0 \longrightarrow \mathbb{R}^0 \to \cdots$$

provides relations defining the subspace of $\mathbb{R}^3$, and furthermore encodes the meta-relations. We call such a sequence a *resolution*.

De Rham cohomology arises naturally from resolving the space of locally constant functions.

To be continued...

References

[1] Iga, K., *What do topologists want from Seiberg-Witten Theory?* arXiv:hep-th/0207271