Problem 1. Using Čech cohomology, give a concise argument that the obstruction for reducing a principal $\mathrm{O}(k)$ bundle P to the subgroup $\mathrm{SO}(k)$ is given by an element $w_1 \in H^1(X;\mathbb{Z}_2)$. Give a formula for a Čech cocycle representing w_1 in terms of the transition functions for P. Finally, argue using the language of Čech cohomology that when $w_1 = 0$, the $\mathrm{SO}(k)$ reductions from a fixed P are parameterized by locally constant $\mathbb{Z}_2$-valued functions.

Remark. The twisted coefficients $\tilde{\mathbb{Z}}$ which appear in non-orientable Poincaré duality are $w_1 \times_{\rho_\pm} \mathbb{Z}$, where $\rho_\pm : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$ is multiplication by ± 1.

Problem 2. A Euclidean metric on $\mathbb{R}^k$ is represented by a positive-definite $k \times k$ matrix M, so that

$$\langle v, v\rangle := v^T M v \quad \text{for } v \in \mathbb{R}^k.$$

If we transform $\mathbb{R}^k$ by $v \mapsto gv$ for $g \in \mathrm{GL}(k)$, how must M transform so that $\langle v, v\rangle$ remains $\mathrm{GL}(k)$-invariant?

Now consider the standard metric $M = \mathrm{Id}_{k\times k}$. Show that the action of $\mathrm{GL}(k)$ on $\mathrm{Id}_{k\times k}$ is well-defined on the cosets $\mathrm{Met}(k) := \mathrm{GL}(k)/\mathrm{O}(k)$. Thus the action of $\mathrm{GL}(k)$ on $\mathrm{Id}_{k\times k}$ induces a map from $\mathrm{Met}(k)$ to positive-definite $k \times k$ matrices. Show that this map is a bijection.

Hint Construct a two-sided inverse from positive-definite matrices to $\mathrm{Met}(k)$.

Remark. If P is a principal $\mathrm{GL}(k)$ bundle, then by Čech theory, a reduction to $\mathrm{O}(k)$ corresponds to a section of $P \times_{\rho_L} \mathrm{Met}(k)$. If $E := P \times_{\rho_{\mathrm{st}}} \mathbb{R}^k$ is the associated vector bundle, then via the bijection above, a section of $P \times_{\rho_L} \mathrm{Met}(k)$ corresponds to a section of the bundle of positive-definite bilinear forms on the fibers of E, i.e. a Euclidean structure on E.

Problem 3. Let X be a closed oriented 4-manifold such that $H_1(X;\mathbb{Z})$ has no 2-torsion (i.e. if $x \in H_1(X;\mathbb{Z})$ satisfies $2x = 0$, then $x = 0$). Show that every $w \in H^2(X;\mathbb{Z}_2)$ is the $\pmod 2$ reduction of some $\tilde{w} \in H^2(X;\mathbb{Z})$.

Hint Use the short exact sequence $0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{\pmod 2} \mathbb{Z}_2 \to 0$ and the associated long exact sequence

$$\cdots \to H^2(X;\mathbb{Z}) \xrightarrow{\pmod 2} H^2(X;\mathbb{Z}_2) \to H^3(X;\mathbb{Z}) \to \cdots.$$

Problem 4. Using the Dold-Whitney theorem, describe all isomorphism classes of $\mathrm{SO}(3)$ principal bundles $[P]$ over $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ in terms of the instanton number $k([P])$ and Stiefel-Whitney class $w_2([P])$.

Next, the Atiyah-Singer index theorem gives the expected dimension for the moduli space of anti-self-dual instantons associated to an isomorphism class $[P]$ of principal G-bundles according to the following formula:

$$\dim \mathscr{M}_{\mathrm{ASD}}([P]) = 4\check{h}(\mathfrak{g})k([P]) - \tfrac{1}{2}(\chi(X) + \sigma(X)) \dim G,$$

where χ and σ denote the Euler characteristic and signature respectively. Given that $\check{h}(\mathfrak{so}(3)) = 2$, compute the expected dimension for each $[P]$.

Hint For any given $w_2([P]) \in H^2(X; \pi_1(\mathrm{SO}(3))) \simeq H^2(X; \mathbb{Z}_2)$, in order to compute $k([P]) \pmod 1$, lift $w_2([P])$ to some $\tilde{w}_2 \in H^2(X; \mathbb{Z})$. The Dold-Whitney theorem then implies that the admissible instanton numbers $k \in \mathbb{Q}$ and Stiefel-Whitney classes $w_2 \in H^2(X; \mathbb{Z}_2)$ are those which satisfy

$$k([P]) \equiv -\tfrac{1}{4} \langle \tilde{w}_2 \smile \tilde{w}_2, [X] \rangle \pmod 1.$$

Problem 5. Give an example of a smooth simply-connected 4-manifold X, together with an isomorphism class $[P]$ of a smooth principal SO(3) bundle over X such that the topological charge of $[P]$ is an integer, but the structure group does not extend to SU(2).

Problem 6. Suppose that $\{f_\alpha\}$ is a partition of unity so that $\sum_\alpha f_\alpha = 1$, and the support of f_α is contained in U_α. Show that the local expression $A_\alpha = -\sum_\gamma f_\gamma d\phi_{\alpha\gamma}\phi_{\alpha\gamma}^{-1}$ satisfies the transformation rule to define a connection A. Show that F_A is flat on any open subset where all but one of the f_α is zero.

Chern-Simons form

Problem 7. Let X be a compact oriented 4-manifold with boundary. (There need not be any metric on X.) Let $P \to X$ be a principal bundle, and $A \in \mathscr{A}_P$.

- Verify that $\Omega^\bullet(X; \mathfrak{g}_{\mathrm{Ad}})$ is a Lie superalgebra, i.e. if $\alpha, \beta, \gamma \in \Omega^\bullet(X; \mathfrak{g}_{\mathrm{Ad}})$, then
$$[\alpha \wedge \beta] = -(-1)^{|\alpha||\beta|} [\beta \wedge \alpha],$$
$$[\alpha \wedge [\beta \wedge \gamma]] = [[\alpha \wedge \beta] \wedge \gamma] + (-1)^{|\alpha||\beta|} [\beta \wedge [\alpha \wedge \gamma]],$$
and in particular, if $a \in \Omega^1(X; \mathfrak{g}_{\mathrm{Ad}})$, then $[a \wedge [a \wedge a]] = 0$.
- Verify that $d_A F_A = 0$.
- For $\alpha \in \Omega^p(X; \mathfrak{g}_{\mathrm{Ad}})$, verify that $d_A d_A \alpha = [F_A \wedge \alpha]$.
- Suppose also $A_0 \in \mathscr{A}_P$. Define $a := A - A_0 \in \Omega^1(X; \mathfrak{g}_{\mathrm{Ad}})$. Verify that
$$F_A = F_{A_0} + d_{A_0} a + \tfrac{1}{2} [a \wedge a], \text{ and}$$
$$d_{A_0} F_A = [F_A \wedge a].$$
- Let $\langle , \rangle_{\mathfrak{g}}$ be an invariant metric on $\mathfrak{g}$ so that $\langle [\chi_1, \chi_2], \chi_3 \rangle_{\mathfrak{g}} = -\langle \chi_2, [\chi_1, \chi_3] \rangle_{\mathfrak{g}}$. Verify that
$$\langle [\alpha \wedge \beta] \wedge \gamma \rangle_{\mathfrak{g}} = -(-1)^{|\alpha||\beta|} \langle \beta \wedge [\alpha \wedge \gamma] \rangle_{\mathfrak{g}}.$$
- For any $\omega, \eta \in \Omega^\bullet(X; \mathfrak{g}_{\mathrm{Ad}})$, verify that
$$d \langle \omega \wedge \eta \rangle_{\mathfrak{g}} = \langle d_A(\omega \wedge \eta) \rangle_{\mathfrak{g}}$$
for any choice of connection A by comparing the expressions in a local frame:
$$\left(d \langle \omega \wedge \eta \rangle_{\mathfrak{g}}\right)_\alpha = d \langle \omega_\alpha \wedge \eta_\alpha \rangle_{\mathfrak{g}} = \langle d(\omega_\alpha \wedge \eta_\alpha) \rangle_{\mathfrak{g}},$$
$$\left(\langle d_A(\omega \wedge \eta) \rangle_{\mathfrak{g}}\right)_\alpha = \left\langle (d\omega_\alpha + [A_\alpha \wedge \omega_\alpha]) \wedge \eta_\alpha + (-1)^{|\omega|} \omega_\alpha \wedge (d\eta_\alpha + [A_\alpha \wedge \eta_\alpha]) \right\rangle_{\mathfrak{g}}.$$

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- Reformulating some of the above identities in more useful form, we have

$$\begin{aligned} d_{A_0} a &= F_A - F_{A_0} - \tfrac{1}{2} [a \wedge a] , \\ d_{A_0} F_{A_0} &= 0, \\ d_{A_0} F_A &= [F_A \wedge a] , \\ d \langle \omega \wedge \eta \rangle_{\mathfrak{g}} &= \langle d_A (\omega \wedge \eta) \rangle_{\mathfrak{g}} . \end{aligned}$$

  Verify that

$$\int_{\partial X} \left\langle a \wedge \left(F_{A_0} + F_A - \tfrac{1}{6} [a \wedge a] \right) \right\rangle_{\mathfrak{g}} = \int_X \langle F_A \wedge F_A - F_{A_0} \wedge F_{A_0} \rangle_{\mathfrak{g}} .$$

  When $\partial X = \emptyset$, conclude that $\int_X \langle F_A \wedge F_A \rangle$ is independent of A.

Hint Recall that for $\omega \in \Omega^{\bullet}(X; \mathfrak{g}_{\mathrm{Ad}})$,

$$(d_A \omega)_\alpha = d\omega_\alpha + [A \wedge \omega_\alpha] .$$

Furthermore, if $\left\{ \chi^i \right\}_{i=1}^{\dim \mathfrak{g}}$ is a local basis of sections of $\mathfrak{g}_{\mathrm{Ad}}$, then $\omega = \sum \chi^i \otimes \omega_i$. Thus

$$\begin{aligned} \omega \wedge \eta &= \sum \chi^i \otimes \chi^j \otimes \omega_i \wedge \eta_j \in \Omega^{|\omega|+|\eta|}(X; \mathfrak{g}_{\mathrm{Ad}} \otimes \mathfrak{g}_{\mathrm{Ad}}), \\ [\omega \wedge \eta] &= \sum \left[\chi^i, \chi^j \right] \otimes \omega_i \wedge \eta_j \in \Omega^{|\omega|+|\eta|}(X; \mathfrak{g}_{\mathrm{Ad}}), \\ \langle \omega \wedge \eta \rangle_{\mathfrak{g}} &= \sum \left\langle \chi^i, \chi^j \right\rangle \omega_i \wedge \eta_j \in \Omega^{|\omega|+|\eta|}(X). \end{aligned}$$

SU(2) **instanton number**

Problem 8. Let $P \to S^4$ be a principal SU(2) bundle. Let B_{α_1} and B_{α_2} be two thickened hemispheres whose interiors U_{α_1} and U_{α_2} cover S^4. For some generic partition of unity $f_{\alpha_1} + f_{\alpha_2} = 1$ with $f_{\alpha_i} \geq 0$ and $\mathrm{supp}(f_{\alpha_i}) \subset U_{\alpha_i}$, consider the connection

$$\left\{ A_{\alpha_1} = -f_{\alpha_2} (d\phi_{\alpha_1\alpha_2}) \phi^{-1}_{\alpha_1\alpha_2}, A_{\alpha_2} = -f_{\alpha_1} (d\phi_{\alpha_2\alpha_1}) \phi^{-1}_{\alpha_2\alpha_1} \right\} .$$

Show that

$$\int_{S^4} \langle F_A \wedge F_A \rangle_{\mathfrak{g}} = \tfrac{1}{6} \int_{\partial B_{\alpha_1}} \langle \gamma \wedge [\gamma \wedge \gamma] \rangle_{\mathfrak{g}} ,$$

where $\gamma = (d\phi_{\alpha_1\alpha_2}) \phi^{-1}_{\alpha_1\alpha_2}$.

Hint In a neighborhood of ∂B_{α_1}, the function f_{α_2} is identically equal to 1. Consequently, $F_A = 0$ near ∂B_{α_1}, and outside of B_{α_1}. Take A_0 to be the trivial connection on B_{α_1}, and use the Chern-Simons form.

Problem 9. Define the right-invariant Maurer-Cartan form $\omega \in \Omega^1(G; \mathfrak{g})$ by

$$\omega \left(\frac{d}{dt}\bigg|_{t=0} e^{t\chi} g_0 \right) = \chi,$$

so that if $\phi : X \to G$, then $\phi^*(\omega) = (d\phi)\phi^{-1}$.

Now suppose that $\phi : S^3 \to \mathrm{SU}(2)$. Since $\mathrm{SU}(2) \simeq S^3$, the homotopy class of ϕ is determined by $\deg \phi \in \mathbb{Z}$, the sign of which depends on the relative orientations of S^3 and $\mathrm{SU}(2)$. The degree can be computed cohomologically as

$$\deg \phi = \frac{\int_{S^3} \phi^*(\nu)}{\int_{\mathrm{SU}(2)} \nu},$$

for any $\nu \in \Omega^3(\mathrm{SU}(2))$ such that $\int_{\mathrm{SU}(2)} \nu \neq 0$. We take $\nu = \langle \omega \wedge [\omega \wedge \omega] \rangle_{\mathfrak{g}}$, for some arbitrary choice of invariant metric $\langle \rangle_{\mathfrak{g}}$.

To compute the denominator, recall that

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \mid |z_1|^2 + |z_2|^2 = 1 \right\}.$$

The standard metric on S^3 has $\mathrm{vol}(S^3) = \frac{1}{2}(2\pi)^2$. By varying z_1 and z_2 by unit vectors along the unit sphere, we see that the orthonormal basis of $T_e\mathrm{SU}(2) = \mathfrak{su}(2)$ corresponding to the standard metric on S^3 is

$$\left\{ \chi_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \chi_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \chi_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}.$$

Thus the metric on SU(2) corresponding to the standard metric on S^3 is $\langle \chi, \xi \rangle_{S^3} := \frac{1}{2}\mathrm{Tr}(\chi \xi^\dagger) = -\frac{1}{2}\mathrm{Tr}(\chi\xi)$ on $\mathfrak{su}(2)$. Let $\left\{ \theta^1, \theta^2, \theta^3 \right\}$ denote the dual basis to $\{\chi_1, \chi_2, \chi_3\}$. Then $\omega = \sum_{i=1}^3 \chi_i \otimes \theta^i$. Show that

$$\langle \omega \wedge [\omega \wedge \omega] \rangle = 6 \langle \chi_1, [\chi_2, \chi_3] \rangle_{\mathfrak{g}} \, \theta^1 \wedge \theta^2 \wedge \theta^3 = -12 \, \theta^1 \wedge \theta^2 \wedge \theta^3.$$

This form is invariant, which you may assume. Using the orientation $\theta^1 \wedge \theta^2 \wedge \theta^3$ on SU(2), conclude that

$$\int_{\mathrm{SU}(2)} \langle \omega \wedge [\omega \wedge \omega] \rangle_{S^3} := -6(2\pi)^2,$$

and

$$\deg \phi_{\alpha_1\alpha_2} = \frac{1}{2}(2\pi)^{-2} \int_X \mathrm{Tr}\left(F_A \wedge F_A \right).$$

This is the expression for the SU(2) instanton number.