Problem Set 1

Due Tuesday, 17 February 2015.

Recall from the conventions section that for vector fields V and W, the second covariant derivative is

$$\nabla^2_{V\otimes W} = \nabla_V \nabla_W - \nabla_{\nabla_V W}.$$

The second term removes the undesired differentiation of W by V.

Problem 1. Write a similar expression for $\nabla^3_{U\otimes V\otimes W}$ in terms of ∇ and ∇^2.

The point of this exercise is to become comfortable with higher covariant derivatives. It's important because curvature is the antisymmetric part of the second covariant derivative.

Recall the claim that a short exact sequence of chain complexes

$$0 \longrightarrow \mathscr{C}^\bullet \xrightarrow{f} \mathscr{D}^\bullet \xrightarrow{g} \mathscr{E}^\bullet \longrightarrow 0$$

induces a long exact sequence on cohomology

$$\cdots \longrightarrow H^p(\mathscr{C}^\bullet, d) \xrightarrow{f} H^p(\mathscr{D}^\bullet, d) \xrightarrow{g} H^p(\mathscr{E}^\bullet, d) \xrightarrow{f^{-1}\circ d\circ g^{-1}} H^{p+1}(\mathscr{C}^\bullet, d) \xrightarrow{f} \cdots$$

Problem 2. Verify that $f^{-1}\circ d\circ g^{-1}$ is well-defined, and that the sequence is exact. If you have already done this before, do not do this problem. If you get the idea and get bored, you can stop early.

You will need to consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{C}^{p+1} & \xrightarrow{f} & \mathscr{D}^{p+1} & \xrightarrow{g} & \mathscr{E}^{p+1} & \longrightarrow & 0 \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \mathscr{C}^{p} & \xrightarrow{f} & \mathscr{D}^{p} & \xrightarrow{g} & \mathscr{E}^{p} & \longrightarrow & 0
\end{array}$$

To begin, consider $x \in H^p(\mathscr{E}^\bullet, d)$. To show that $f^{-1}(d(g^{-1}x))$ is well-defined, we must show that it corresponds to a unique $y \in H^{p+1}(\mathscr{C}^\bullet, d)$. By definition, $x = [e]$ for some $e \in \mathscr{E}^p$ such that $de = 0$. The rows of this diagram are exact, so g is surjective, so $e = gd$ for some $d \in \mathscr{D}^p$...

You will make choices along the way, but you must show that any two answers $c, c' \in \mathscr{C}^{p+1}$ satisfy $[c] = [c'] \in H^{p+1}(\mathscr{C}^\bullet, d)$.

You can find the proof in any homological algebra textbook. However, you should try to figure it out on your own and instead ask friends for help when you get stuck. This exercise will require you to become comfortable with equivalence relations.

This construction is important since we will develop the theory of principal bundles from this homological perspective. This will allow us to write down a few short exact sequences and easily conclude that, for instance, an oriented real vector bundle $E \to X$ admits a spin structure iff $w_2(E) = 0$.