Funny Functions

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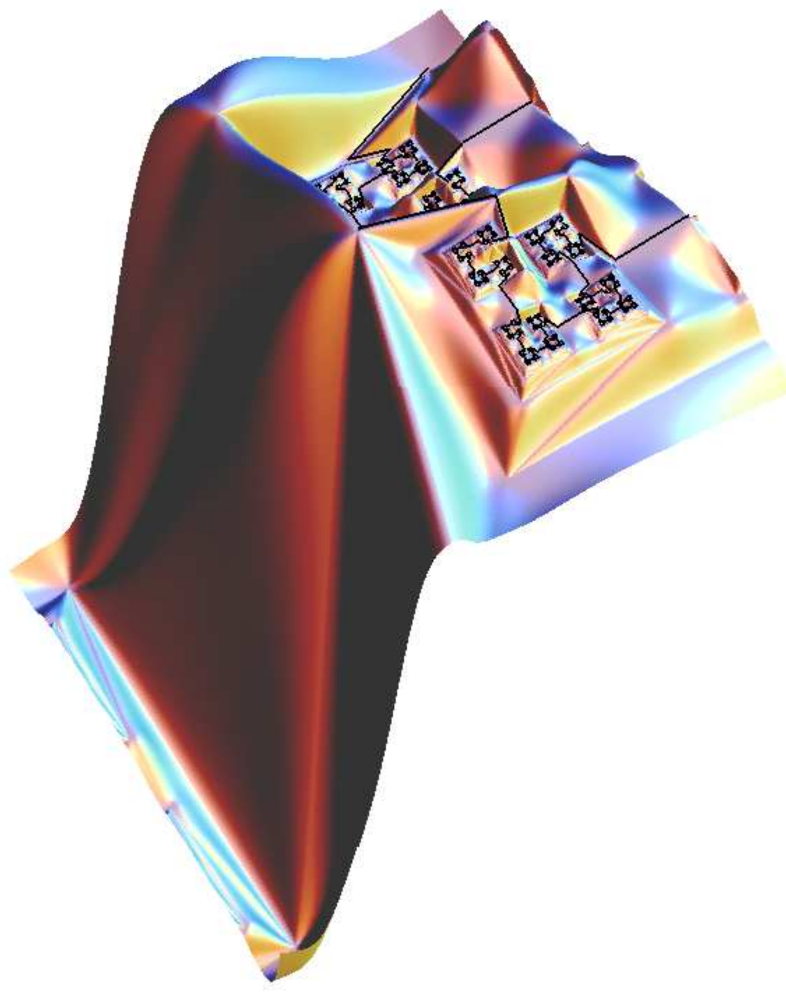
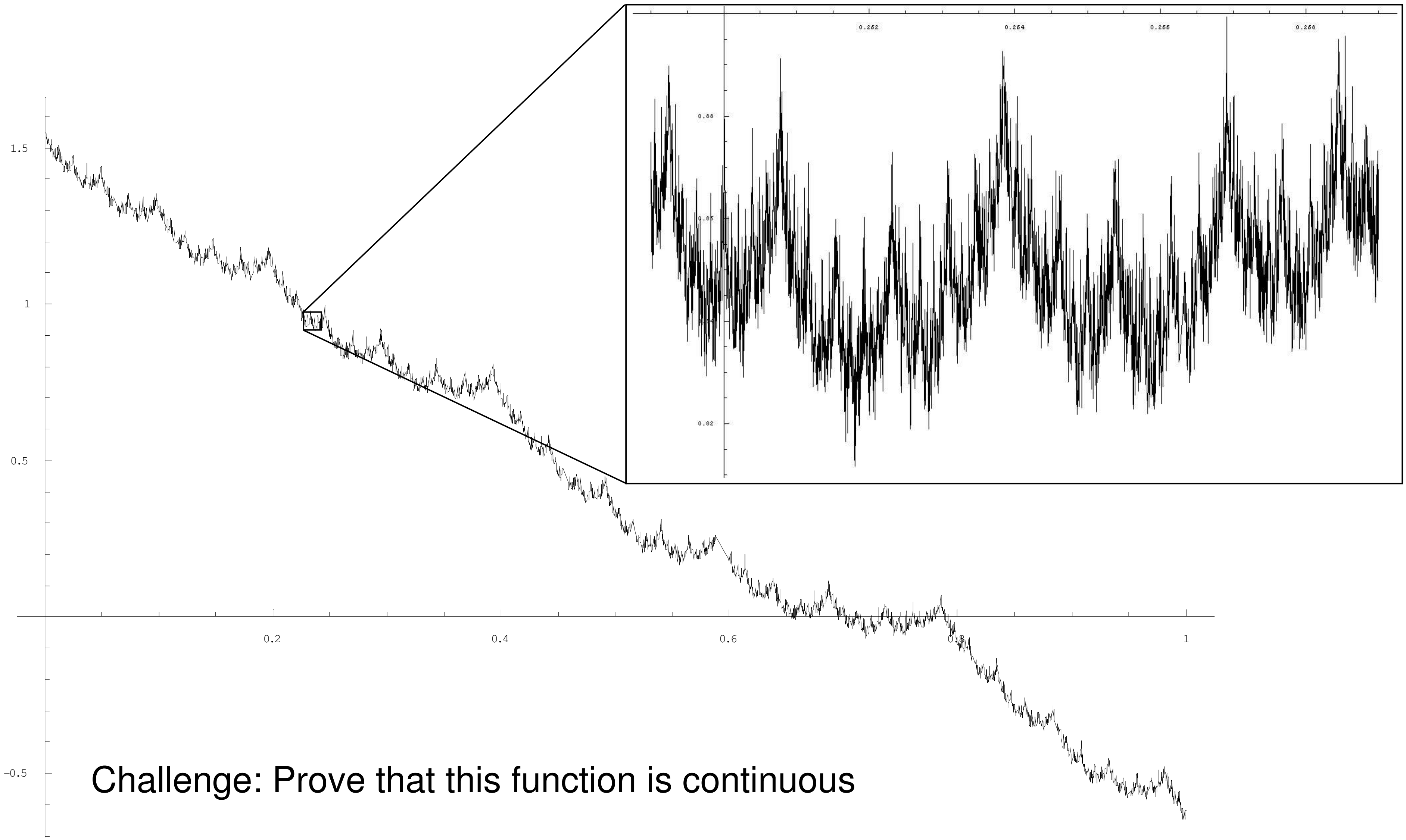
(MIT)



Working definition of continuity:
Zoom in on the domain and the range gets arbitrarily small

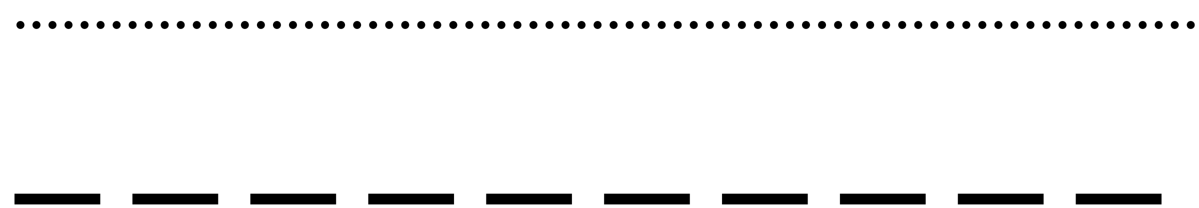
```
f[x_] := Sum[Cos[2^n x] / n^2, {n, 1, 1000}]
```



Challenge: Prove that this function is continuous

Everywhere Discontinuous

$$f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$$

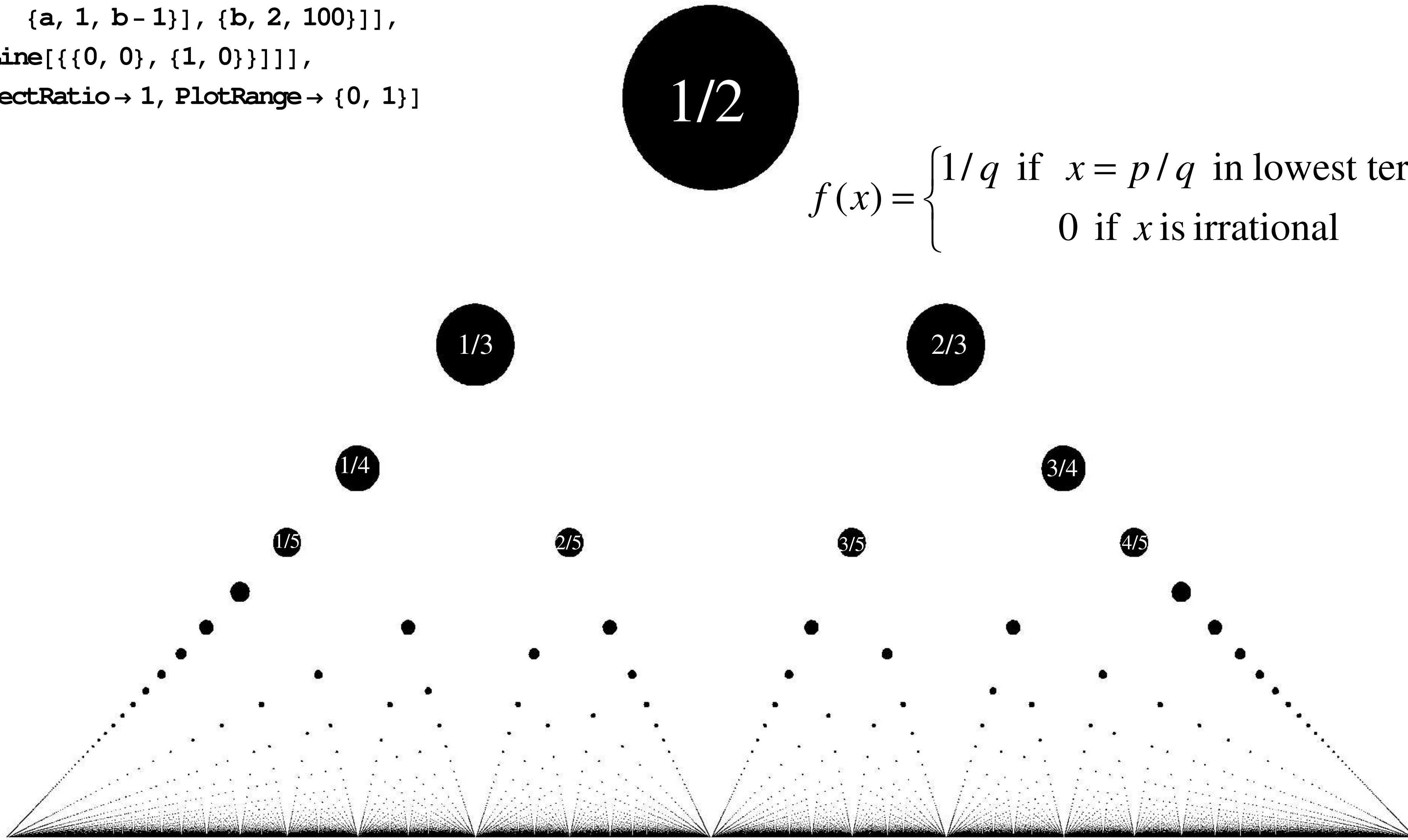


Lame schematic graph. We can do better! (later)

Continuous at every irrational number
Discontinuous at every rational number

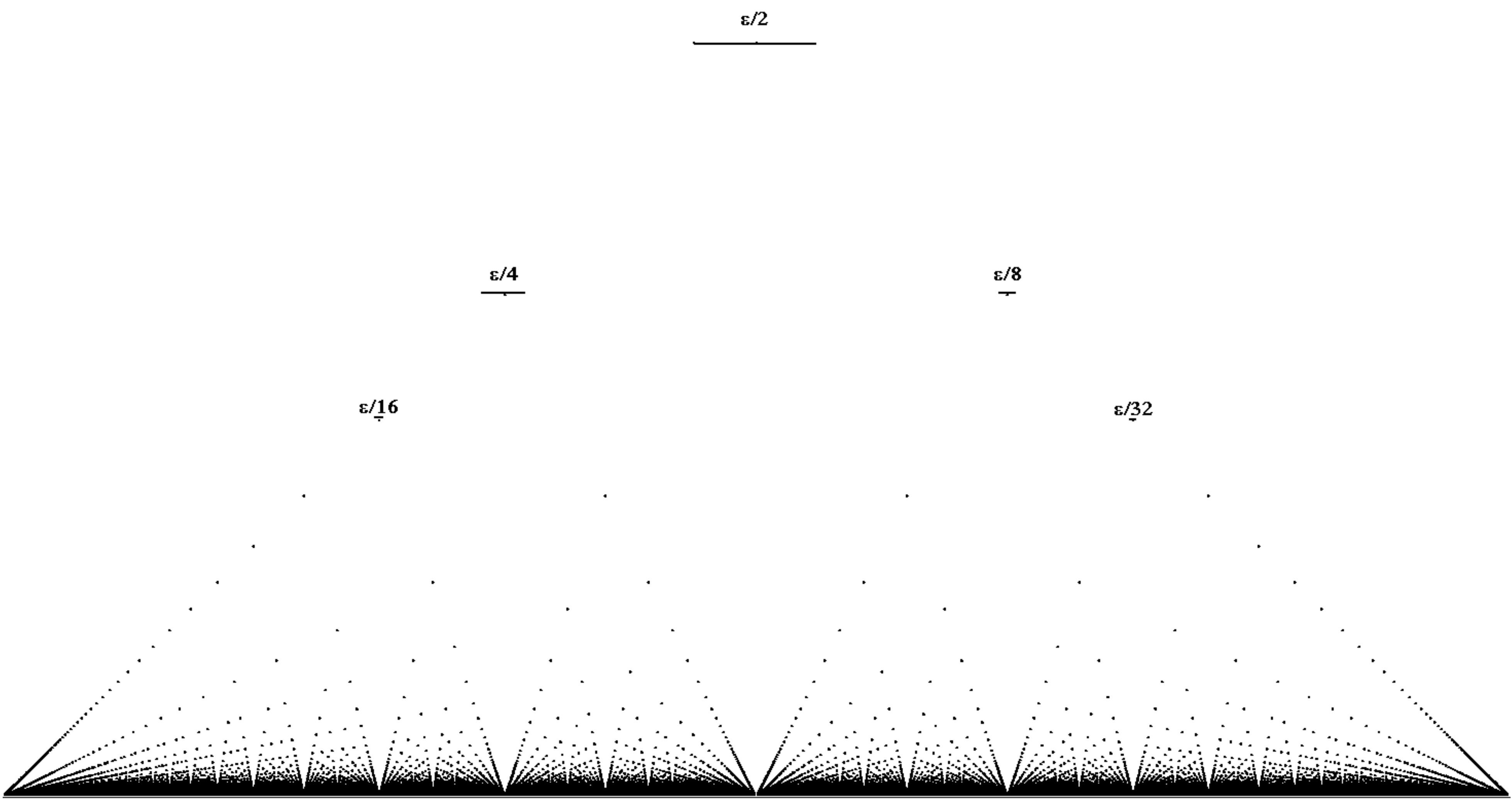
```
Show[Graphics[Append[Flatten[
    Table[Table[
      Disk[{a / b, GCD[a, b] / b}, 1 / b^2 / 4],
      {a, 1, b - 1}], {b, 2, 100}]],
   Line[{{0, 0}, {1, 0}}]]],
 AspectRatio → 1, PlotRange → {0, 1}]
```

$$f(x) = \begin{cases} 1/q & \text{if } \; x = p/q \text{ in lowest terms} \\ 0 & \text{if } \; x \text{ is irrational} \end{cases}$$

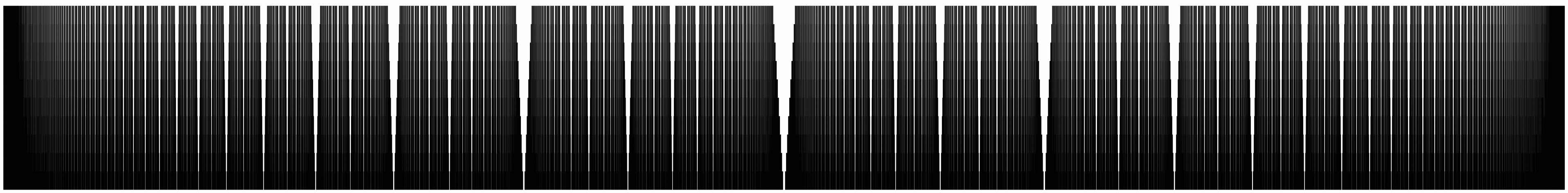


What is the width of the rational numbers?

Overestimate the width as ε

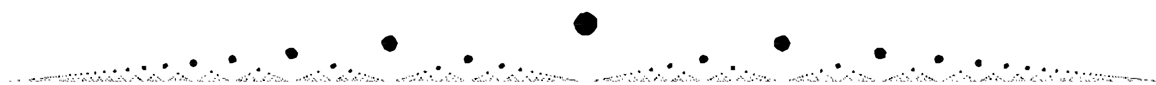


The “negligible” holes left in the irrational numbers



Everywhere Discontinuous

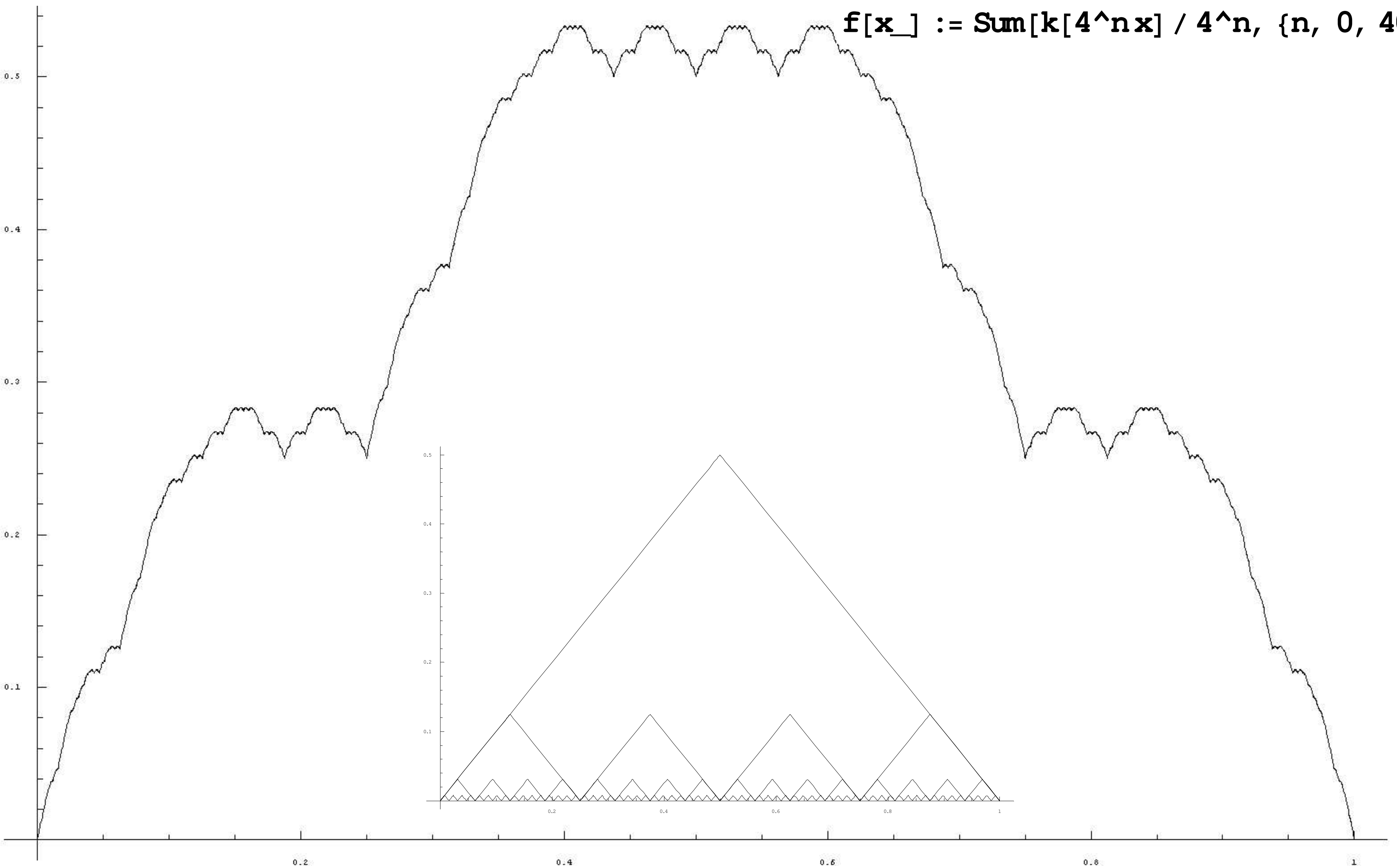
$$f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$$



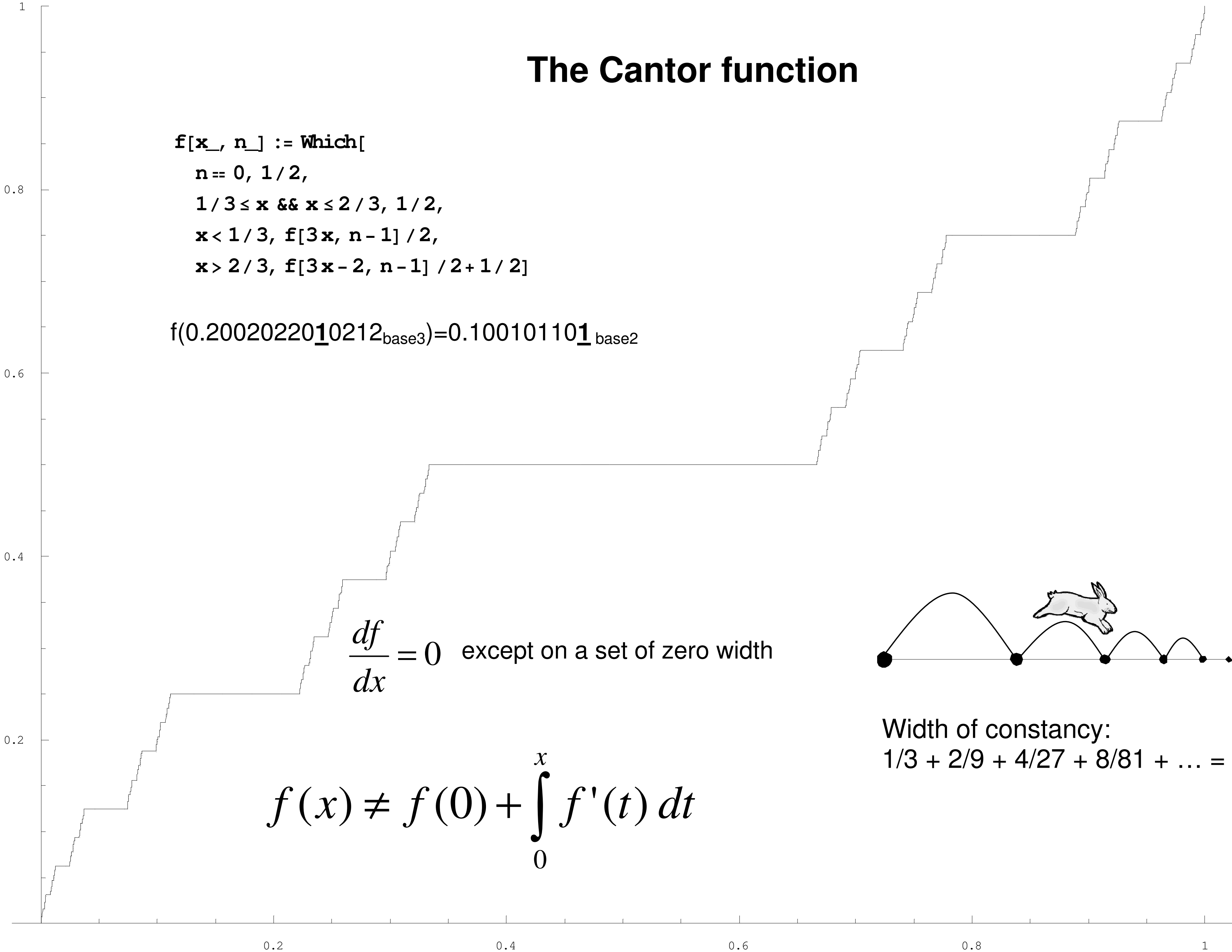
Much more satisfying

Nowhere differentiable function

```
j[x_] := If[x < 1 / 2, x, 1 - x]
k[x_] := j[FractionalPart[x]]
f[x_] := Sum[k[4^n x] / 4^n, {n, 0, 40}]
```



The Cantor function




```
f[x_, n_] := Which[
  n == 0, 1/2,
  1/3 ≤ x && x ≤ 2/3, 1/2,
  x < 1/3, f[3x, n-1]/2,
  x > 2/3, f[3x-2, n-1]/2 + 1/2]
```

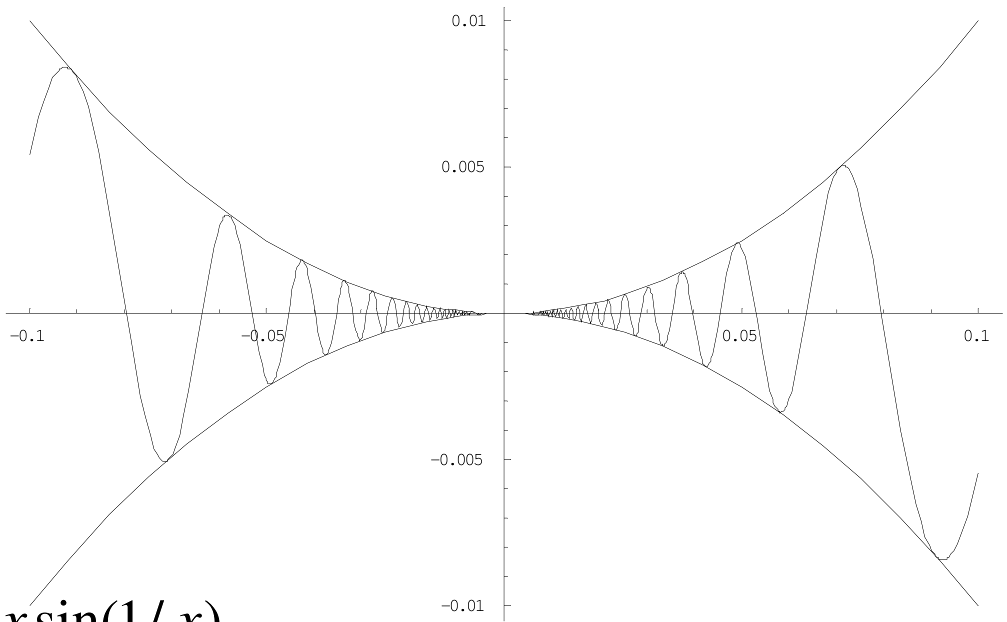
$f(0.20020220\underline{\mathbf{1}}0212_{base3})=0.10010110\underline{\mathbf{1}}_{base2}$

$$\frac{df}{dx}=0$$ except on a set of zero width

$$f(x) \neq f(0)+\int_0^x f'(t)\,dt$$

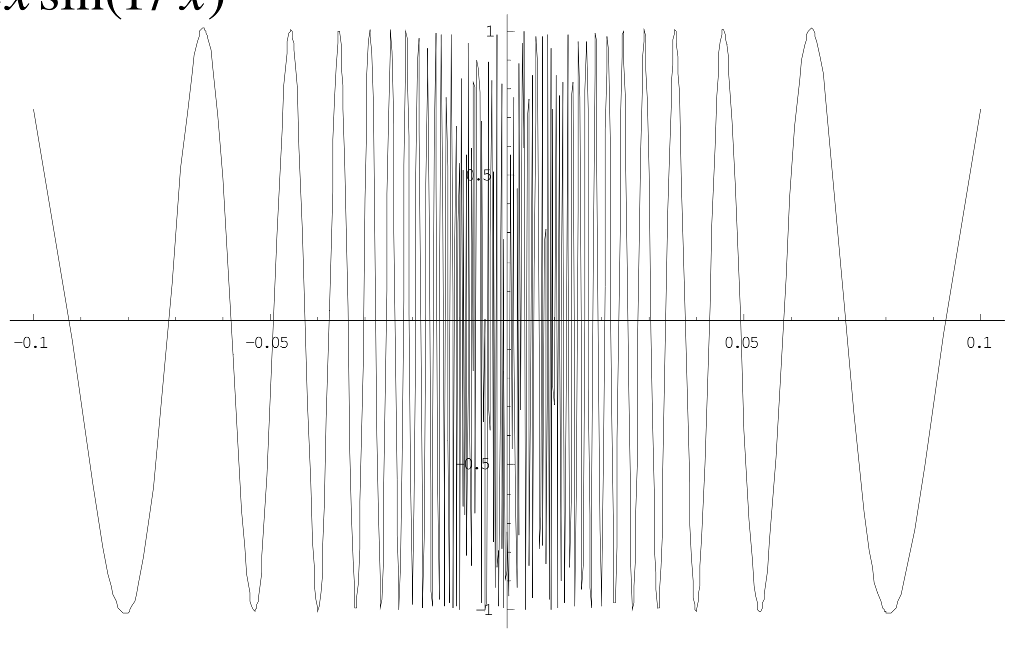
Width of constancy:
1/3 + 2/9 + 4/27 + 8/81 + … = 1.

First derivative exists everywhere but is discontinuous

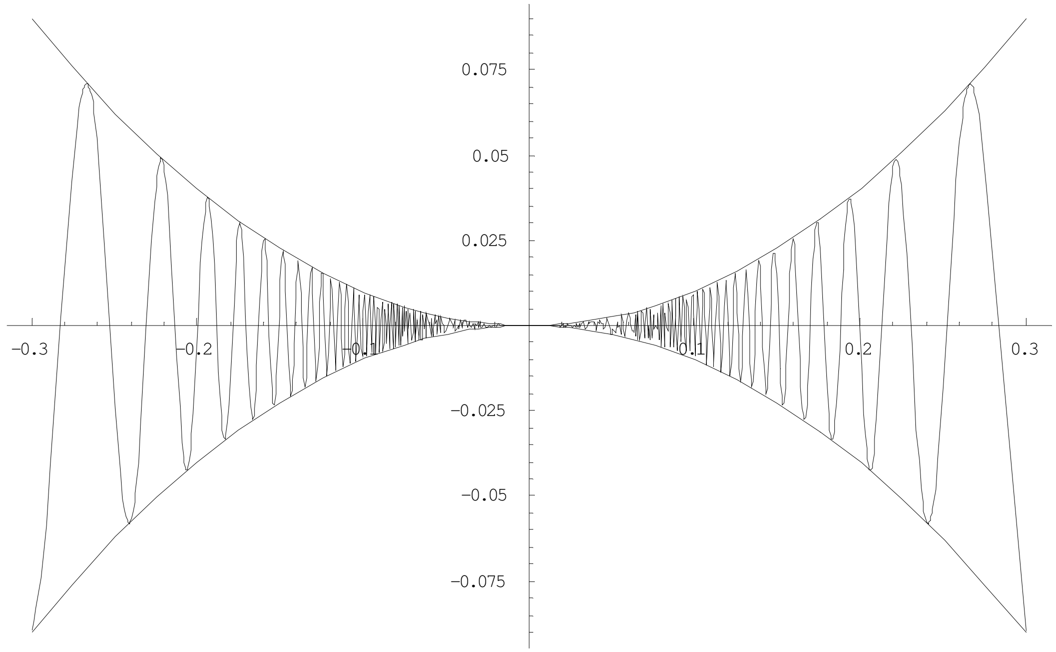


$$f(x) = x^2 \sin(1/x)$$

$$f'(x) = -\cos(1/x) + 2x\sin(1/x)$$

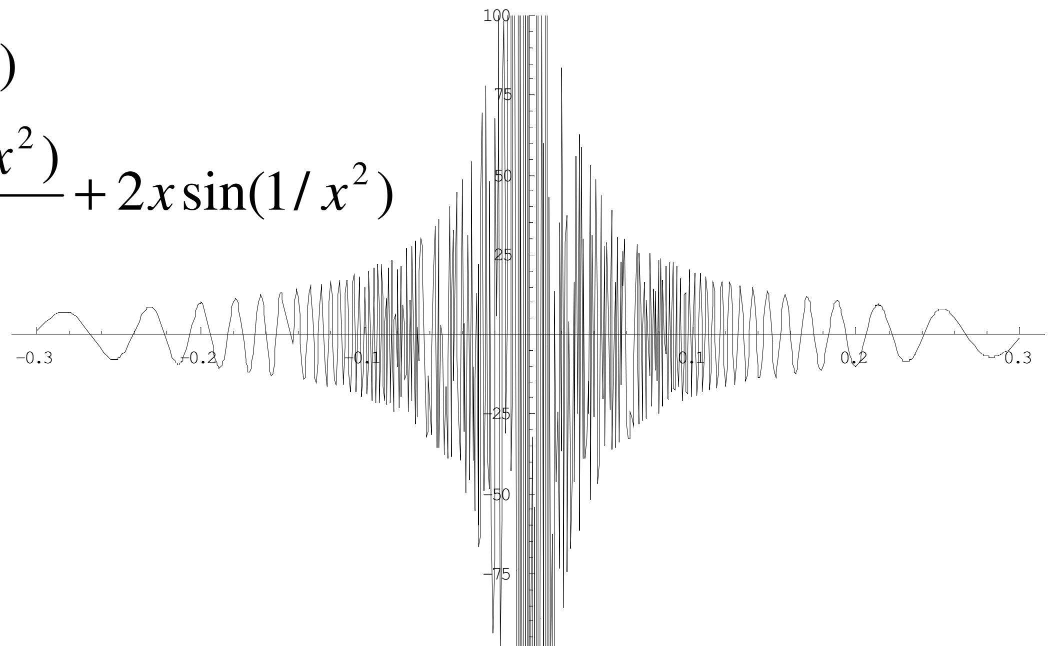


Derivative exists everywhere but is unbounded near zero

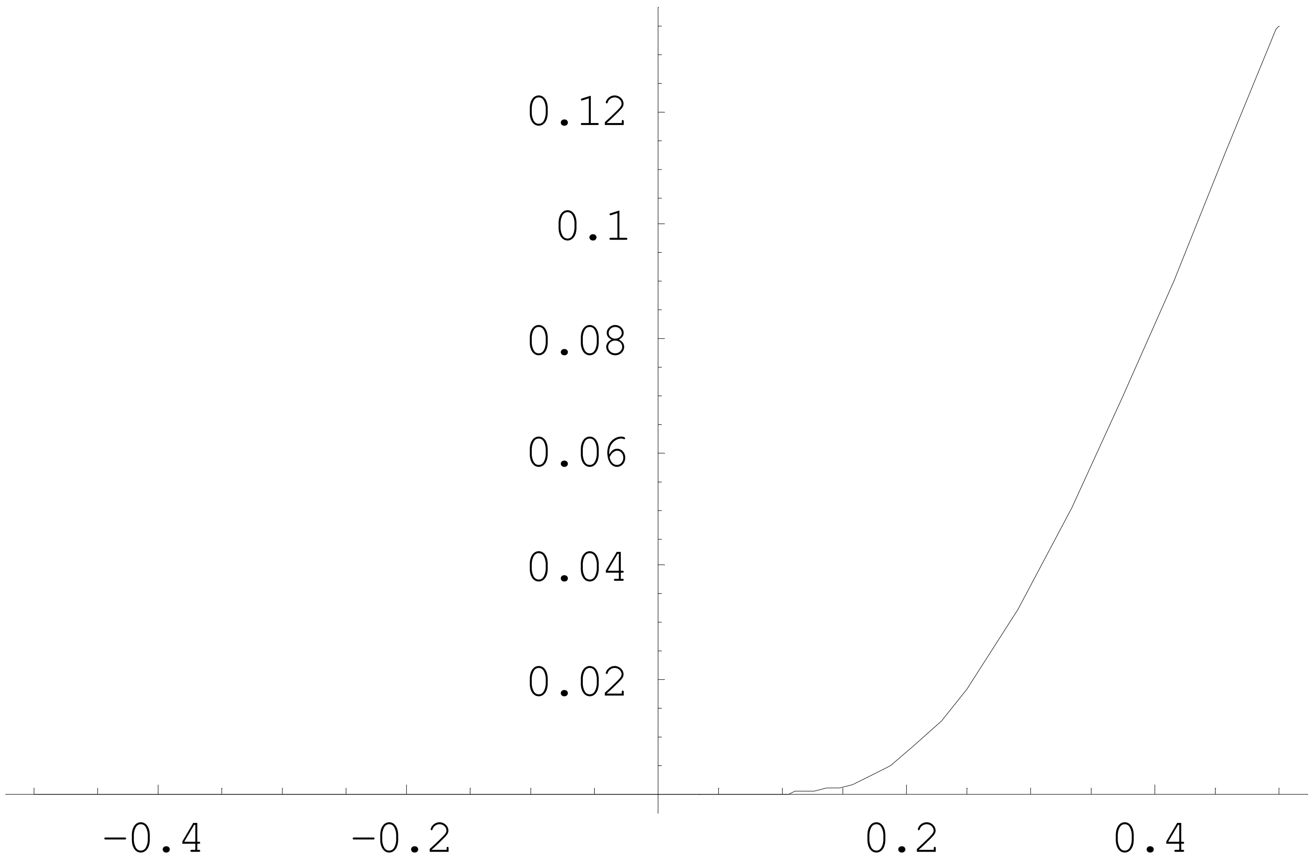


$$f(x) = x^2 \sin(1/x^2)$$

$$f'(x) = \frac{-2\cos(1/x^2)}{x} + 2x\sin(1/x^2)$$



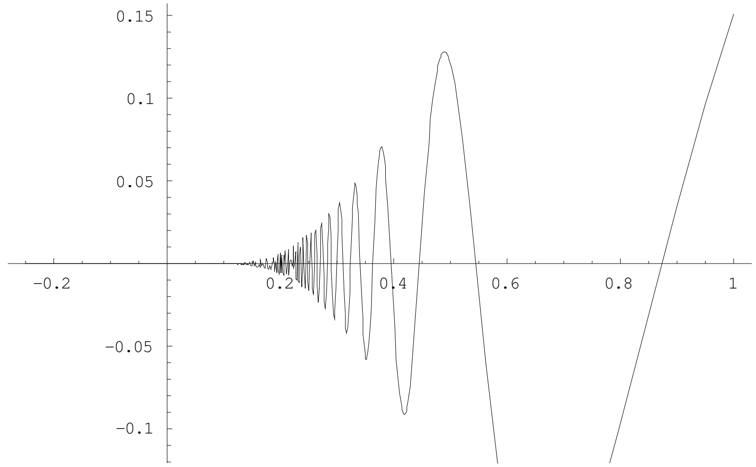
Nonzero smooth function with all derivatives vanishing at a point

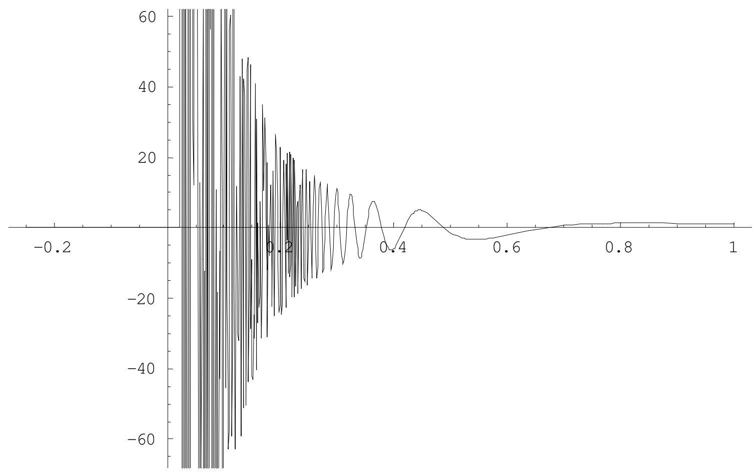


$$f(x) = \begin{cases} \exp(-1/x) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Has a zero of infinite order.

Has a Taylor approximation at each point, yet the first derivative is discontinuous and unbounded

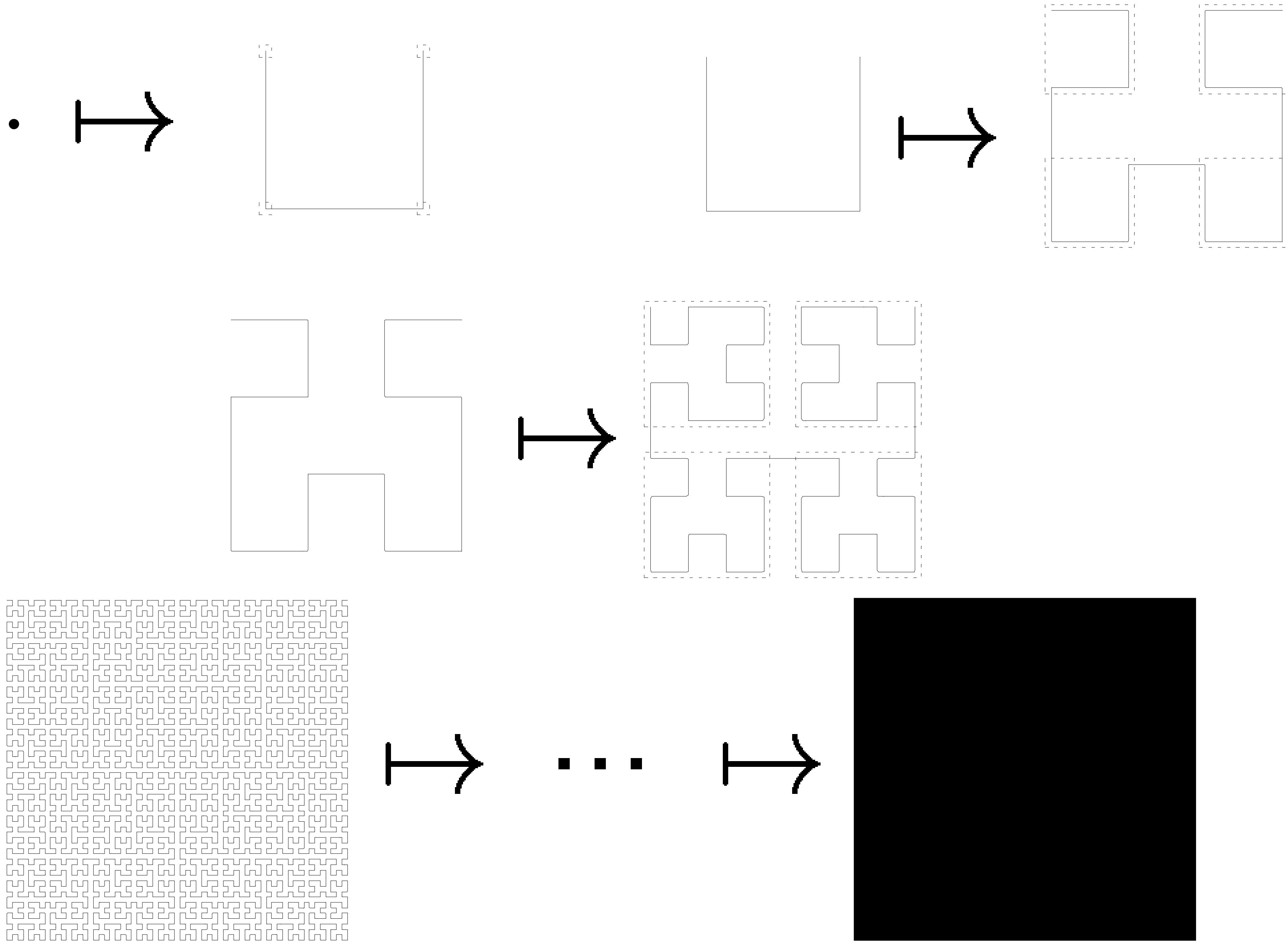




$$f(x) = \begin{cases} \exp(-1/x)\sin(\exp(1/x)) & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

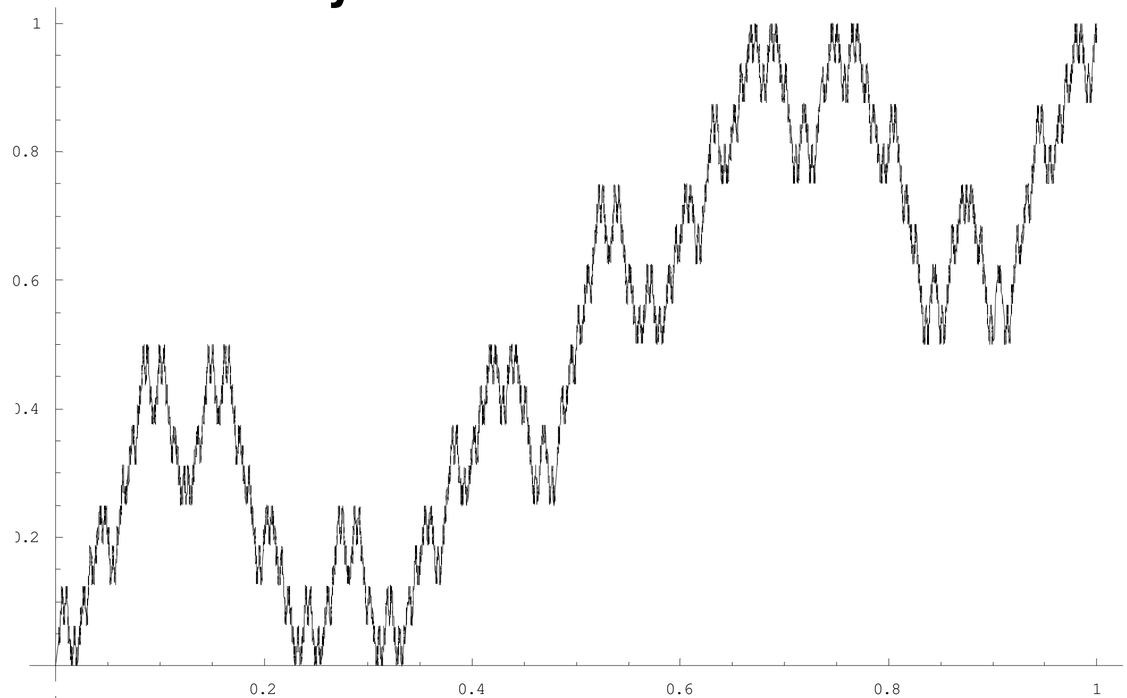
$$f'(x) = \begin{cases} x^{-2}(\exp(-1/x)\sin(\exp(1/x)) - \cos(\exp(1/x))) & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

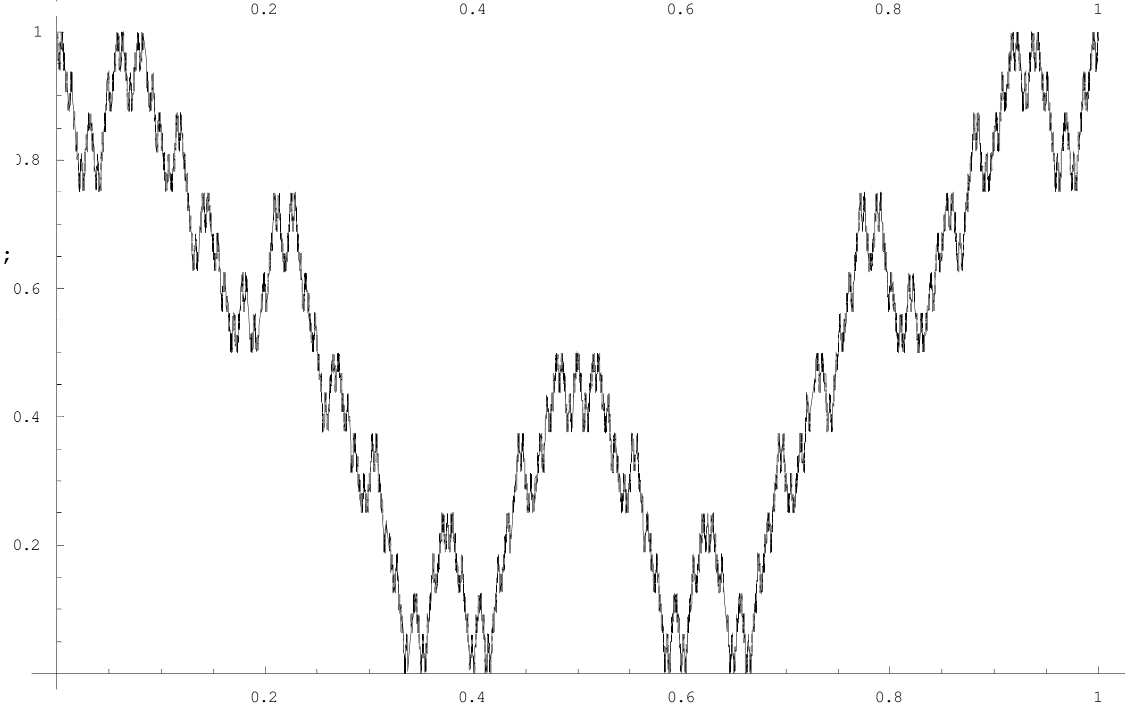
Peano curve



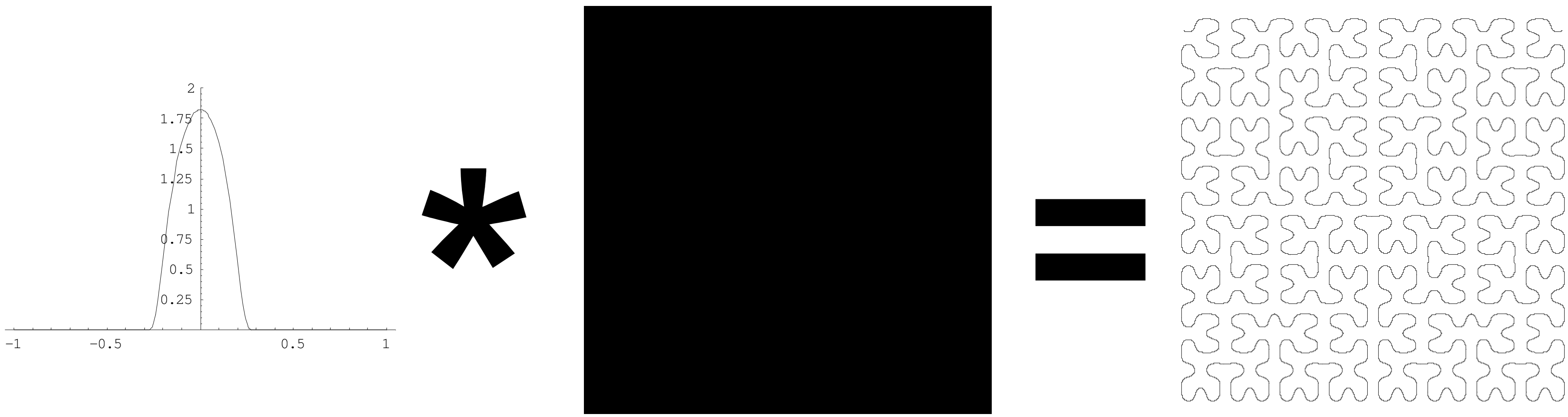
Parameterization by Area

```
x[t_, n_] := Which[n == 0, 1 / 2,
   0 ≤ reparam[t, n, 2] ≤ 1, 2^-(n + 1),
   0 ≤ reparam[t, n, 4] ≤ 1, 1 / 2 - 2^-(n + 1) + 2^-n reparam[t, n, 4],
   0 ≤ reparam[t, n, 6] ≤ 1, 1 - 2^-(n + 1),
   0 ≤ reparam[t, n, 1] ≤ 1, (1 - y[reparam[t, n, 1], n - 1]) / 2,
   0 ≤ reparam[t, n, 3] ≤ 1, x[reparam[t, n, 3], n - 1] / 2,
   0 ≤ reparam[t, n, 5] ≤ 1, (1 + x[reparam[t, n, 5], n - 1]) / 2,
   0 ≤ reparam[t, n, 7] ≤ 1, (1 + y[reparam[t, n, 7], n - 1]) / 2];
y[t_, n_] := Which[n == 0, 1 / 2,
   0 ≤ reparam[t, n, 2] ≤ 1, 1 / 2 + 2^-(n + 1) - 2^-n reparam[t, n, 2],
   0 ≤ reparam[t, n, 4] ≤ 1, 1 / 2 - 2^-(n + 1),
   0 ≤ reparam[t, n, 6] ≤ 1, 1 / 2 - 2^-(n + 1) + 2^-n reparam[t, n, 6],
   0 ≤ reparam[t, n, 1] ≤ 1, 1 - x[reparam[t, n, 1], n - 1] / 2,
   0 ≤ reparam[t, n, 3] ≤ 1, y[reparam[t, n, 3], n - 1] / 2,
   0 ≤ reparam[t, n, 5] ≤ 1, y[reparam[t, n, 5], n - 1] / 2,
   0 ≤ reparam[t, n, 7] ≤ 1, (1 + x[reparam[t, n, 7], n - 1]) / 2];
reparam[t_, n_, i_] := ((4^n - 1) t - Floor[(i - 1) / 2]
     - Floor[i / 2] * (4^(n - 1) - 1)) / If[Mod[i, 2] == 0, 1, 4^(n - 1) - 1];
```

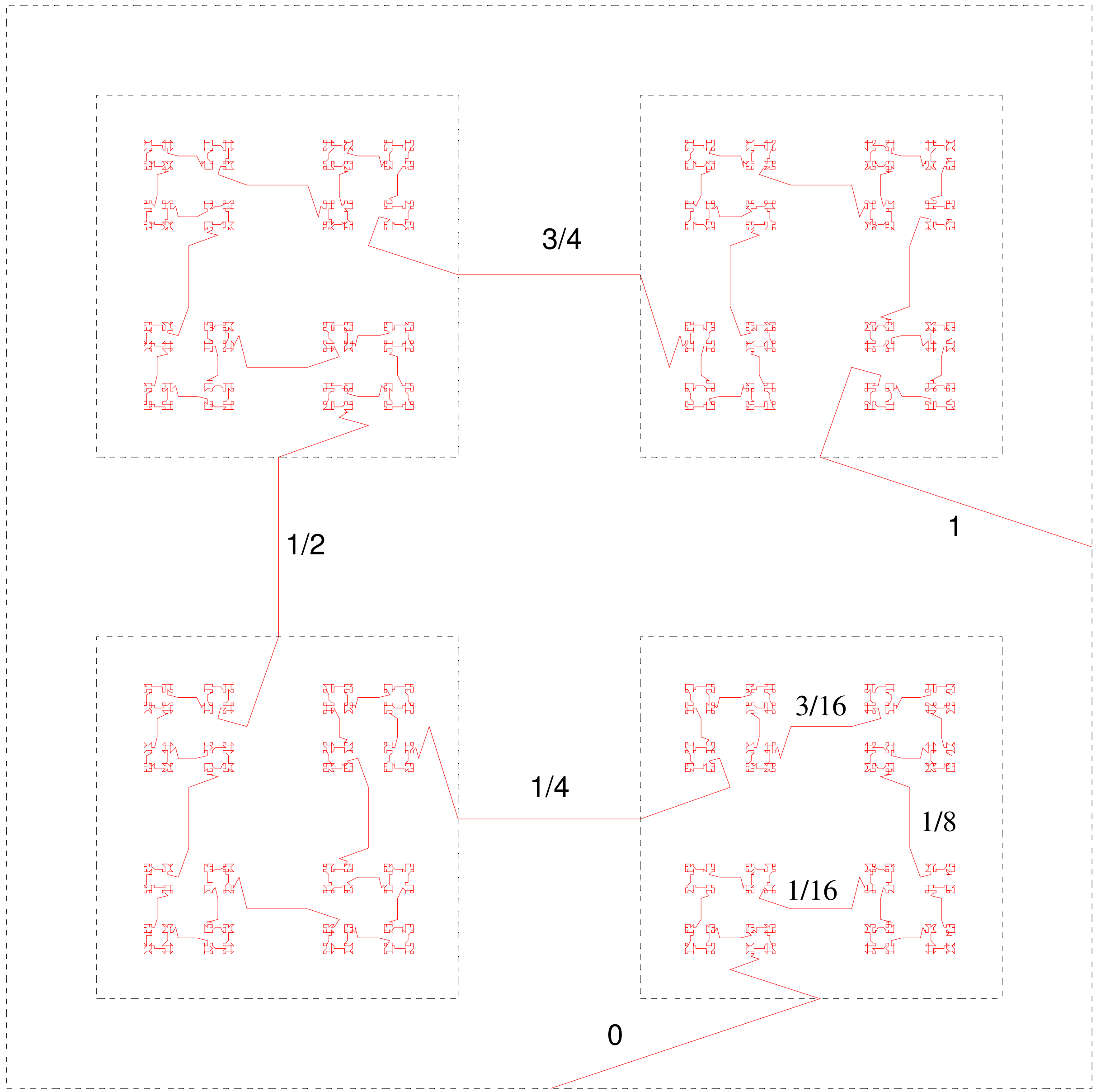




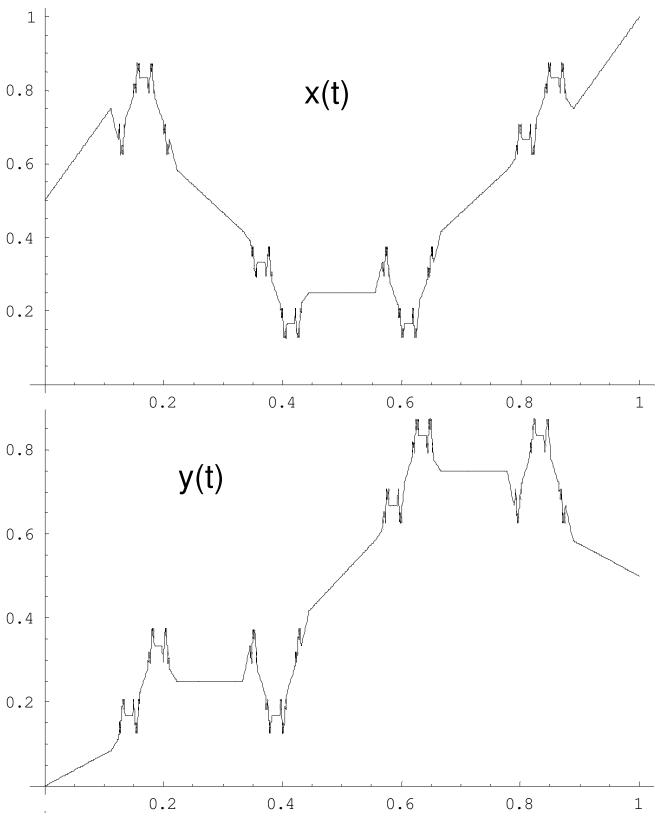
Mollification / Smoothing / Convolution / Averaging

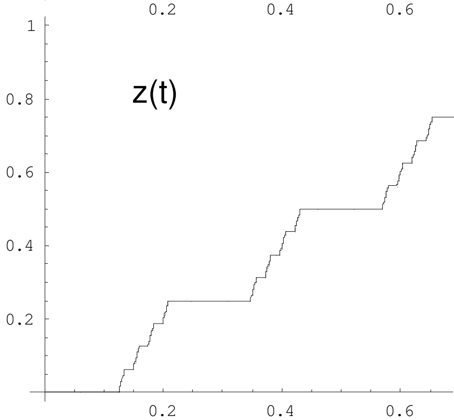


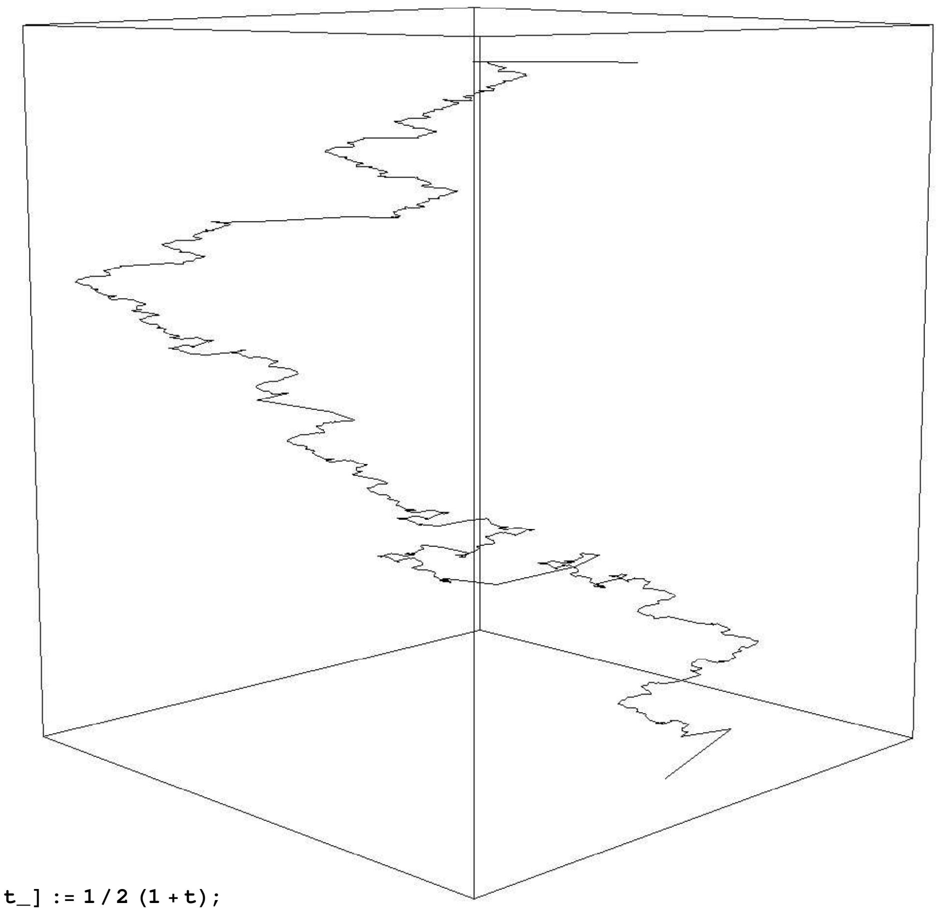
Whitney's curve



(Bad) parameterization of Whitney's curve







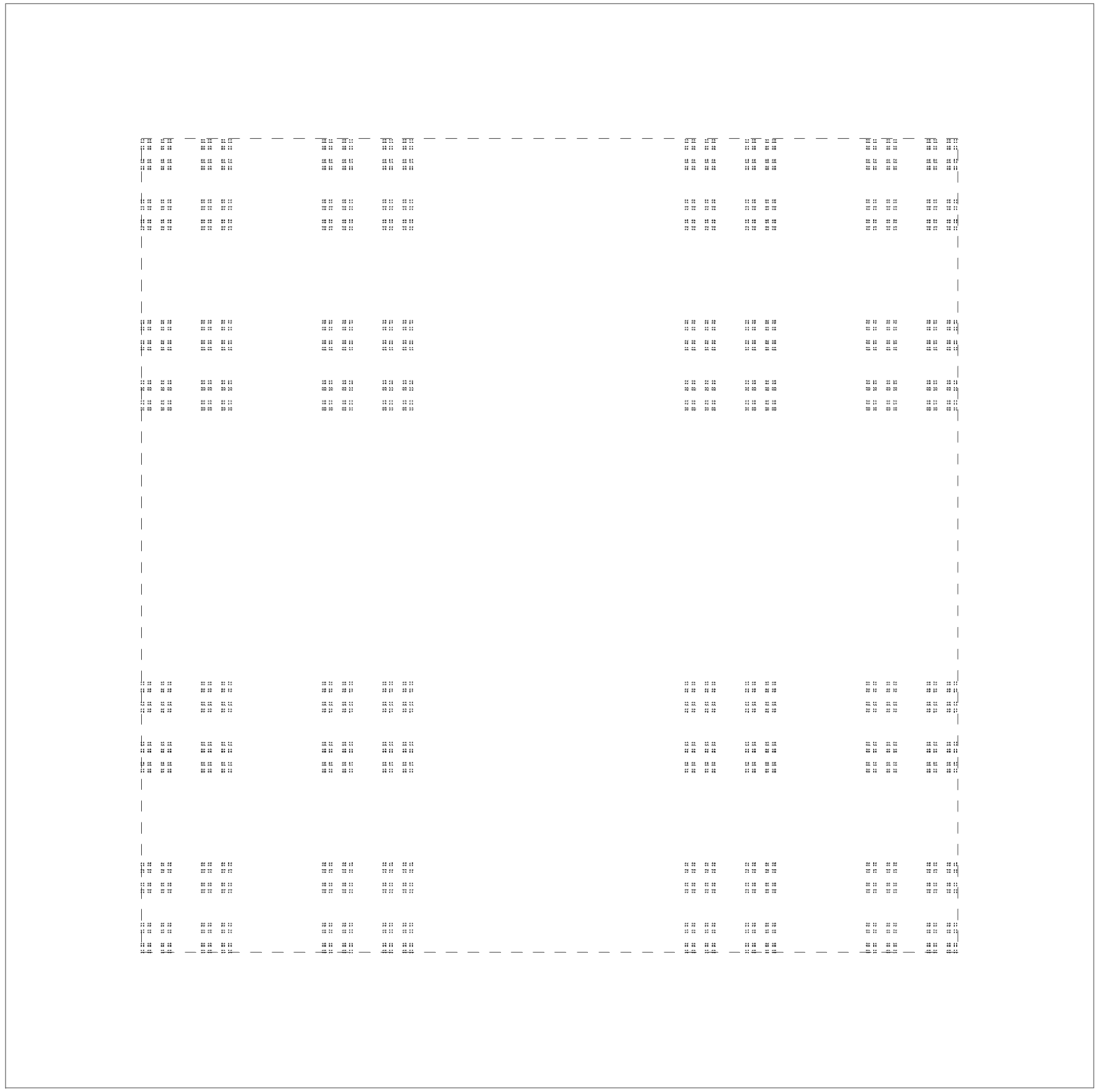
```
x[0, t_] := 1 / 2 (1 + t);
y[0, t_] := t / 2;
z[0, t_] := t;
```

```
x[i_, t_] := Which[
   0 ≤ 9 t ≤ 1, 1 / 2 + 1 / 4 * 9 t,
   2 ≤ 9 t ≤ 3, 7 / 12 - 1 / 6 (9 t - 2),
   4 ≤ 9 t ≤ 5, 1 / 4,
   6 ≤ 9 t ≤ 7, 5 / 12 + 1 / 6 (9 t - 6),
   8 ≤ 9 t ≤ 9, 3 / 4 + 1 / 4 (9 t - 8),
   1 ≤ 9 t ≤ 2, 11 / 12 - 1 / 3 x[i - 1, 9 t - 1],
   3 ≤ 9 t ≤ 4, 5 / 12 - 1 / 3 y[i - 1, 9 t - 3],
   5 ≤ 9 t ≤ 6, 1 / 12 + 1 / 3 x[i - 1, 9 t - 5],
   7 ≤ 9 t ≤ 8, 7 / 12 + 1 / 3 y[i - 1, 9 t - 7]
  ];
```

```
y[i_, t_] := Which[
   0 ≤ 9 t ≤ 1, 9 t / 12,
   2 ≤ 9 t ≤ 3, 1 / 4,
   4 ≤ 9 t ≤ 5, 5 / 12 + 1 / 6 (9 t - 4),
   6 ≤ 9 t ≤ 7, 3 / 4,
   8 ≤ 9 t ≤ 9, 7 / 12 - 1 / 12 (9 t - 8),
   1 ≤ 9 t ≤ 2, 1 / 12 + 1 / 3 y[i - 1, 9 t - 1],
   3 ≤ 9 t ≤ 4, 1 / 12 + 1 / 3 x[i - 1, 9 t - 3],
   5 ≤ 9 t ≤ 6, 7 / 12 + 1 / 3 y[i - 1, 9 t - 5],
   7 ≤ 9 t ≤ 8, 11 / 12 - 1 / 3 x[i - 1, 9 t - 7]
  ];
```

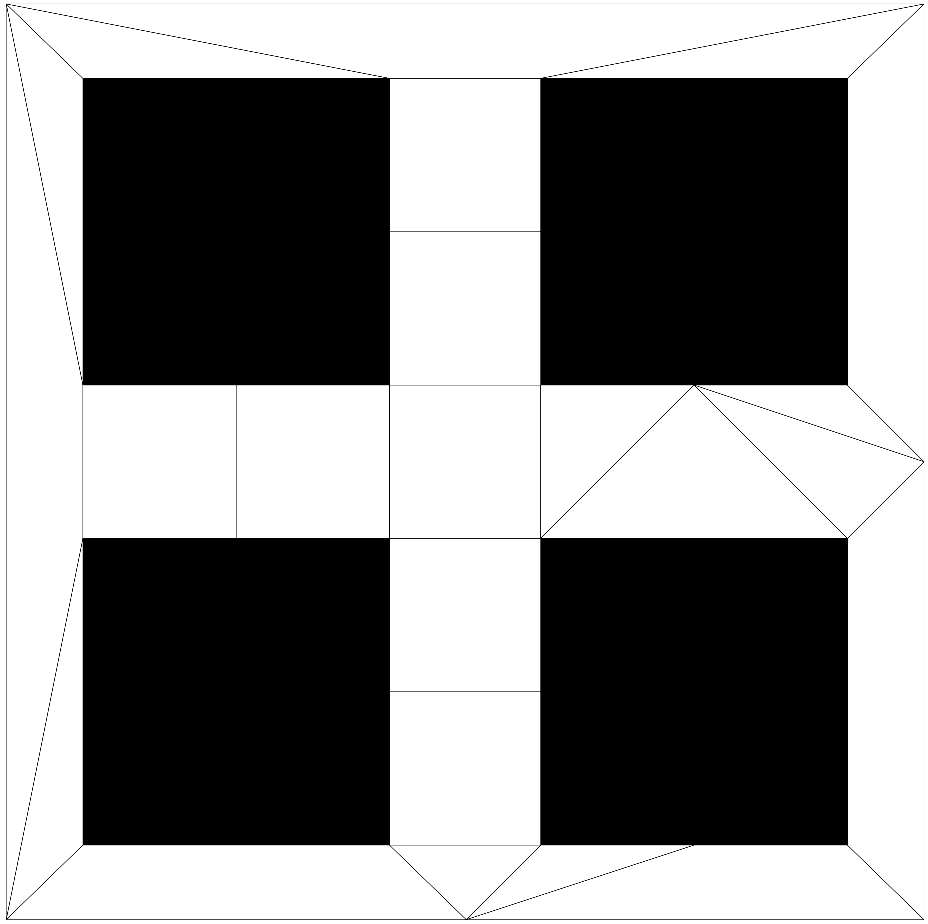
```
z[i_, t_] := Which[
   0 ≤ 9 t ≤ 1, 0,
   2 ≤ 9 t ≤ 3, 1 / 4,
   4 ≤ 9 t ≤ 5, 1 / 2,
   6 ≤ 9 t ≤ 7, 3 / 4,
   8 ≤ 9 t ≤ 9, 1,
   1 ≤ 9 t ≤ 2, 1 / 4 z[i - 1, 9 t - 1],
   3 ≤ 9 t ≤ 4, 1 / 4 + 1 / 4 z[i - 1, 9 t - 3],
   5 ≤ 9 t ≤ 6, 1 / 2 + 1 / 4 z[i - 1, 9 t - 5],
   7 ≤ 9 t ≤ 8, 3 / 4 + 1 / 4 z[i - 1, 9 t - 7]
  ];
```

“Funny” points of Whitney’s curve



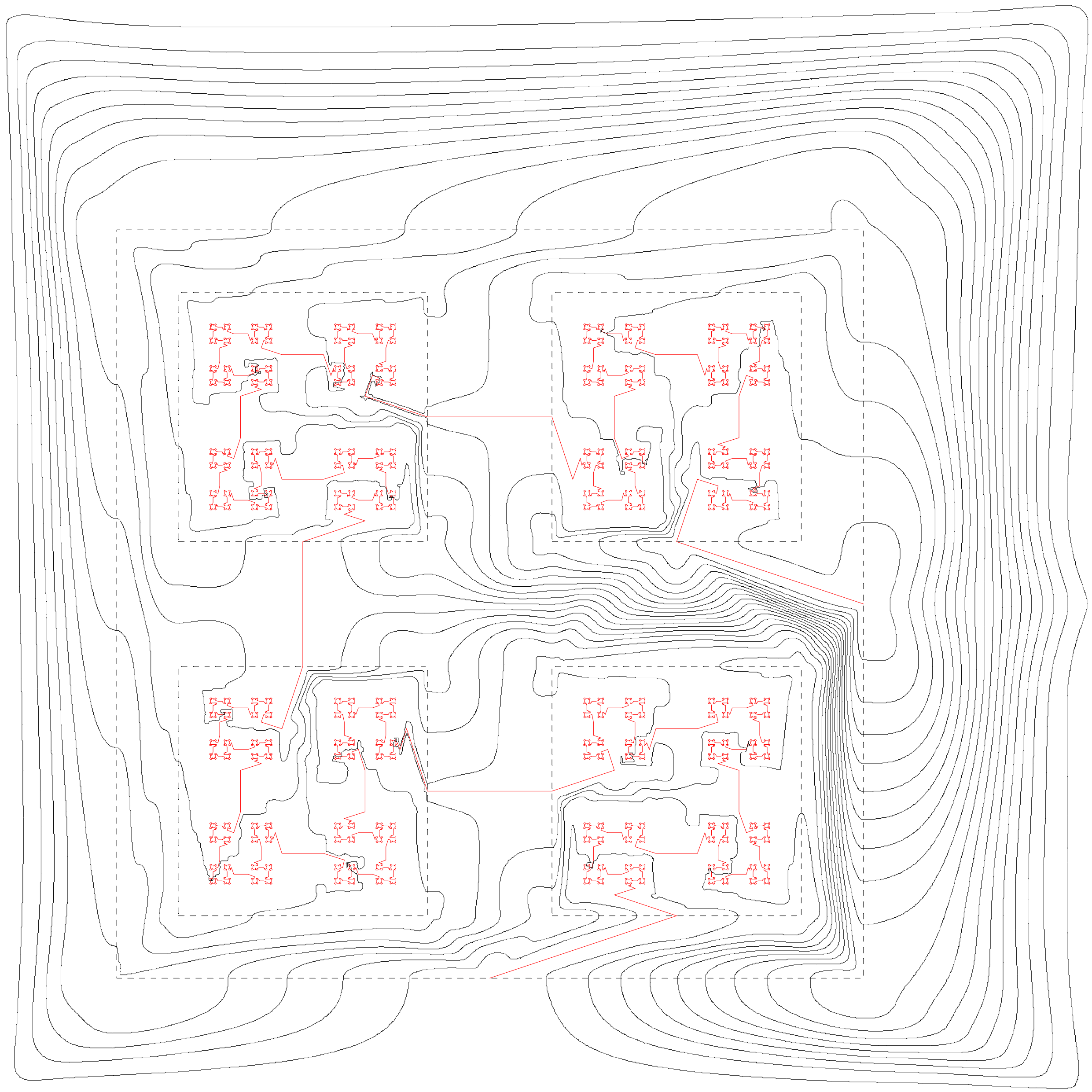
(Cartesian product of two Cantor sets)

Building the mountain

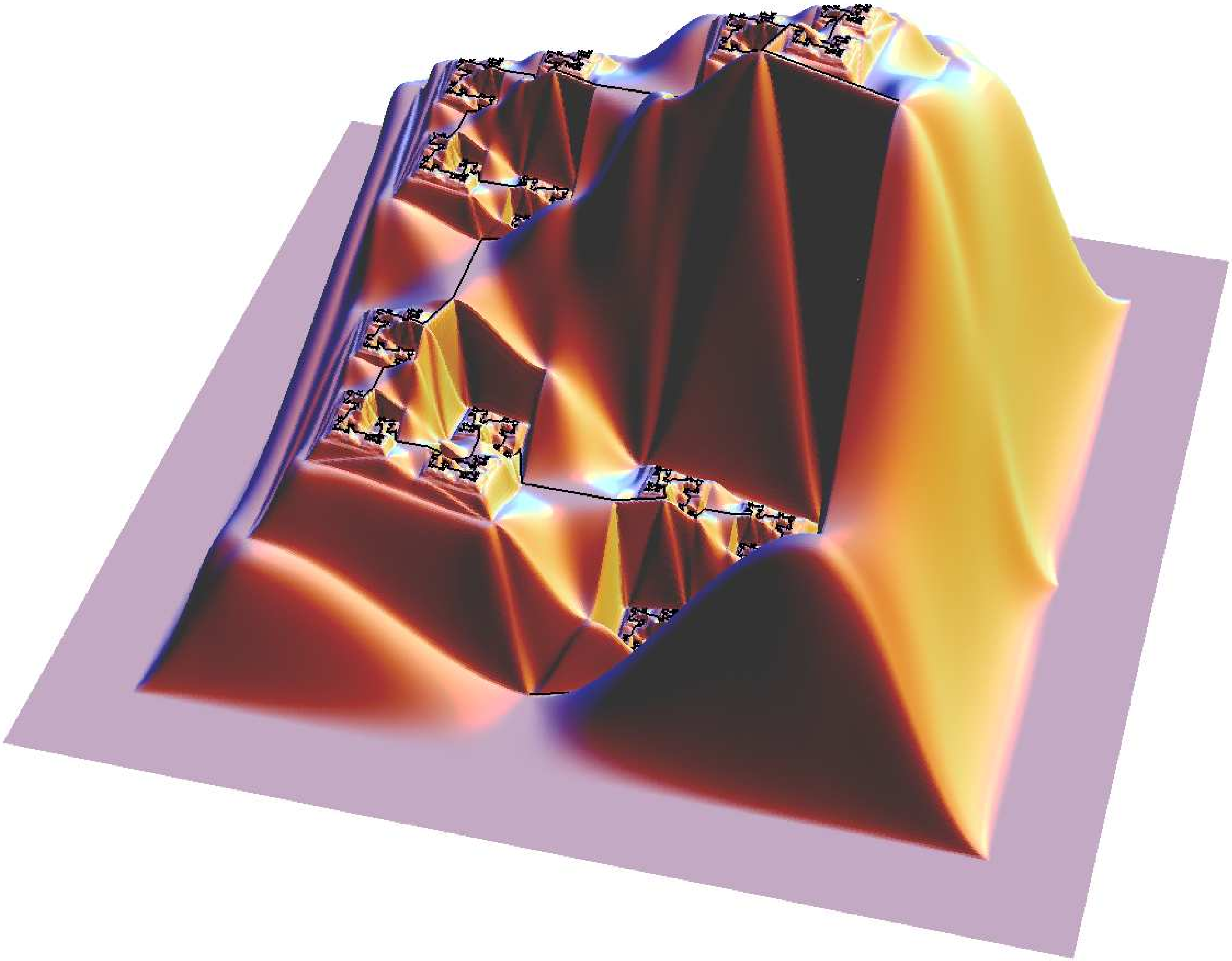


1. Assign values to the nodes
2. Interpolate values along edges in a standard way
3. Fill in the faces with a functions such that along the edges, the gradient is parallel to the edges
4. Define the function recursively inside the black squares: the gradient will match up.

Whitney's Mountain: Topo map



Whitney's Mountain



References

1. Me! (http://tensorial.com)
2. Gelbaum, Olmsted, *Counterexamples in Analysis*
3. H. Whitney, *A Function Not Constant On A Connected Set of Critical Points*. Duke Math. J. 1 (1935), 514-517