$\mathcal{A}^p_1$, $\frac{n}{2}<p<n$ $\quad$ ∂X can be nonempty.

$S_A(\varepsilon)=\{A+a \mid d_A^* a=0,\ *a|_{\partial X}=0,\ \|a\|_{L^n}<\varepsilon\}$.

$m: \left(\mathcal{G}^p_2 \times S_A(\varepsilon)\right)/\mathrm{Stab}(A)$ is diffeo onto image.

$m: \mathcal{G}^p_2 \times S_A(\varepsilon)/\mathrm{Stab}(A) = \mathcal{U}(\varepsilon)$.

For δ small enough, $B_A(\delta) \subset \mathcal{U}(\varepsilon) \subset \mathcal{A}^p_1$

$B_A(\delta) = \{A \mid A+a,\ \|a\|_{L^n}<\delta\}$.

$\mathcal{U}(\varepsilon) \cap B_A(\delta) = W(\delta)$.

To show $W(\delta)$ is a component of $B_A(\delta) \Rightarrow W(\delta)=B_A(\delta)$.

$W(\delta)$ is an intersection of open sets, so it is open.

To show: $W(\delta)$ is closed in $B(\delta)$. If $u(A+b) \in A+a \in S_A(\varepsilon)$

then (E1) $\|a\|_{L^n} \ll 1 \Rightarrow \|d_A u\|_{L^n} + \|a\|_{L^n} \le K\|b\|_{L^n}$.

(E2) $\|a\|_{L^n}, \|b\|_{L^n} \ll 1 \Rightarrow \|d_A u\|_{L^p_1} + \|a\|_{L^p_1} \le K\|b\|_{L^p_1}$, $\frac{n}{2}<p<n$.

$\{A_i = A+b_i\} \subset W(\delta)$ and $b_i \xrightarrow{L^p_1} b$ $\|b\|_{L^n}<\delta$, then $A+b \in W(\delta)$.

$\exists\, u_i, a_i$ so that $u_i \cdot (A+a_i) = A+b_i$, $\|a_i\|_{L^n}<\varepsilon$.

(E1) $\Rightarrow \|d_A u_i\|_{L^n} + \|a_i\|_{L^n} \ll 1$

(E2) $\Rightarrow \|d_A u_i\|_{L^p_1} + \|a_i\|_{L^p_1} \ll 1$, $\frac{n}{2}<p<n$

Pass to a weakly convergent subsequence

$a_i \xrightarrow{L^n_1} a \Rightarrow a_i \xrightarrow[n]{L^n} a$

$u_i \xrightarrow{L^p_2} u \Rightarrow u_i \xrightarrow{L^n_1} u \quad u_i \xrightarrow{C^0} u$

$d_A^* a_i = 0 \Rightarrow d_A^* a = 0$, $\|a\|_{L^n} \le \varepsilon$ using E1 again $\|a\|_{L^n} \le K\delta < \varepsilon$.

Need to check $A+a \sim A+b$.

$$u_i \cdot (A+a_i) = b_i \Rightarrow d_A u_i = u_i b_i - a_i u_i \qquad b_i \rightarrow b$$
$$\downarrow L^n \qquad\qquad \downarrow L^n \qquad\qquad a_i \rightarrow a$$
$$d_A u = u b - a u \qquad\qquad b_i \xrightarrow{C^0} u$$

If A is a C^∞-connection, then $\exists \delta > 0$ so that $\forall$ $A+b \in \tilde{B}_A(\delta)$ in $\mathcal{A}^n$, then $\exists u \in \mathcal{G}^n_1$ so that $u \cdot (A+b) \in \tilde{S}_A(\varepsilon) = \{A+a \mid d_A^* a = 0, \|a\|_{L^n} < \varepsilon\}$. ~~$\exists$ can~~.

Proof: Can approximate $A+b$ by $A+b_i$, b_i are all smooth (say) $\|b_i\|_{L^n} < 2\delta$. Now $A+b_i$ are all gauge-equivalent into $S_A(\varepsilon)$ by the previous result. $\exists$ u_i, a_i so that $u_i(A+a_i) = A+b_i$,

$\|d_A u_i\|_{L^n} + \|a_i\|_{L^n} \le K \|b_i\|_{L^n} \quad \Rightarrow a_i \xrightarrow{L^n} a$ with $\|a\|_{L^n} < \varepsilon$

$u_i \xrightarrow{L^n_1} u \Rightarrow u_i \xrightarrow{L^q} u \ \forall q < \infty$

$$d_A u_i = u_i b_i - a_i u_i \qquad u_i \xrightarrow{L^q} u \ \forall q < \infty$$
$$b_i \xrightarrow{L^n} b \Rightarrow u_i b_i \xrightarrow{L^{n-\varepsilon}} ub$$

$b_i \downarrow L^n\ b$, $a_i \downarrow L^n\ a$, $u_i b_i \downarrow L^n\ \psi$, $a_i u_i \downarrow L^n\ \psi$

$$d_A u = \psi$$

$u_i b \xrightarrow{L^{n-\varepsilon}} ub$

$a_i u_i \xrightarrow{L^{n-\varepsilon}} au$

Theorem: $\exists \eta_0$ so that $\forall A \in \mathcal{A}^2_1(B^4)$ with $\int_{B^4} |F_A|^2 < \eta$,

$\Rightarrow \exists u \in \mathcal{G}^2_2$, $\Gamma + a \in \mathcal{A}^2_1(B^4)$ with $d^*_A a = 0$, $*a|_{\partial B^4} = 0$ and so that $u(\Gamma + a) = A$ and $\int |\nabla_\Gamma a|^2 + |a|^2 \le K \int |F_A|^2$.

Proof: $V(\eta) = \{ A \in \mathcal{A}^4_1(B^4) \mid \int_{B^4} |F_A|^2 < \eta \}$. and this exists;

Observation: $V(\eta)$ is contractible. $i_\lambda : B^4_{1/\lambda} \to B^4_1$, $x \mapsto \lambda x$

$$A_\lambda = i_\lambda^*(A)\big|_{B^4(1)} \quad \text{for } \lambda \le 0$$

$A_\lambda \longrightarrow \Gamma$ on B^4 since $\|a_\lambda\|_{L^4(B_1)} = \|a\|_{L^4(B_\lambda)}$.

$\int_{B^4} |F_{A_\lambda}|^2 \le \int |F_A|^2 < \eta \Rightarrow A \to A_\lambda$ is a def. retraction of $V(\eta)$ to a point.

$\tilde{U}(\varepsilon) = m(\mathcal{G}^4_1 \times \mathcal{S}^2_A(\varepsilon))$ Is this open?

Aside:

Claim: For ε small enough, this is an open set in $\mathcal{A}^4(B^4)$.
Claim deferred.

To show: $\tilde{U}(\varepsilon) \cap V(\eta)$ is closed in $V(\eta)$.

$\Gamma + b_i \in \tilde{U}(\varepsilon) \cap V(\eta)$ and $\Gamma + b_i \xrightarrow{L^4} \Gamma + b \in V(\eta)$.

$\exists \underset{L^4_1}{u_i}, \underset{L^2_2}{a_i}$ $\quad u_i \cdot (\Gamma + a_i) = \Gamma + b_i$, $\|a\|_{L^4} < \varepsilon$, $d^* a_i = 0$, $*a_i|_{\partial B} = 0$

$$\Rightarrow \int_B |F_{\Gamma + a_i}|^2 < \eta \Rightarrow \int |\nabla_\Gamma a_i|^2 + |a_i|^2 \le K \int |F_{\Gamma + a_i}|^2 < K\eta.$$

We can pass to a weakly L^2_1 convergent sequence $a_i \xrightarrow{L^2_1} a$, $u_i \xrightarrow{L^2_2} u$

$u \cdot (\Gamma + a) = \Gamma + b$, $d^*_\Gamma a = 0$, $*a|_{\partial B} = 0$, and if η small enough, $K\eta < \varepsilon$, $\|a\|_{L^4} < \varepsilon$

$\Rightarrow \tilde{U}(\varepsilon) \cap V(\eta)$ is closed in $V(\eta)$.

$$\star a|_\partial = 0$$

$$C\|a\|^2_{L^2_1(B)} \le \int_{B^4} |\nabla_\Gamma a|^2 = \int_{B^4} |(d+d^*) a|^2 = \int_{B^4} |da|^2 = \int |F_{\Gamma + a} - a \wedge a|^2$$

$$\le \int |F_A|^2 + \|a\|^2_{L^4} (\|a\|_{L^2_1})^2.$$

Rearrange.

We already have Palais–Smale sequence $A_i \in \mathcal{A}_{L^2_1}(\Sigma)$

1) $\int_\Sigma |F_{A_i}|^2 < \eta$, 2) $d^*_{A_i} F_{A_i} \xrightarrow{L^2} 0$ $\Rightarrow$ on sufficiently small balls B_α,

$\exists\, u_{i,\alpha}, a_{i,\alpha}$ $u_{i\alpha}(\Gamma + a_{i\alpha}) = A_i|_{B_\alpha}$, $a_{i,\alpha} \xrightarrow{L^2_2} a_\alpha$ on B_α.

$g_{i,\beta\alpha} = (u_{i,\beta}^{-1} u_{i,\alpha})^{\pm 1} \leftarrow$ convention?

$$g_{i,\beta\alpha}(\Gamma + a_{i,\alpha}) = \Gamma + a_{i,\beta}.$$

$\Rightarrow g^{-1}_{i,\beta\alpha} dg_{i,\beta\alpha} + g^{-1}_{i,\beta\alpha} a_{i,\alpha} g_{i,\beta\alpha} = a_{i,\beta}$.

$$dg_{i,\beta\alpha} = g_{i,\beta\alpha} a_{i,\beta} - a_{i,\alpha} g_{i,\beta\alpha}$$

$$\begin{array}{cc} \downarrow L^2_2 & \downarrow L^2_2 \\ a_\alpha & a_\beta \end{array} \quad \text{on } B_\alpha \cap B_\beta$$

$dg_{i,\beta\alpha}$ bounded in L^p $\forall p$. $\Rightarrow g_{i,\beta\alpha} \xrightarrow{L^p_1} g_{\beta\alpha} \Rightarrow g_{i,\beta\alpha} \xrightarrow{L^q} g_{\beta\alpha}$ $\forall q$

$\Rightarrow dg_{i,\beta\alpha} \xrightarrow{L^p} dg_{\beta\alpha}$ $\forall p$

$\Rightarrow g_{i,\beta\alpha} \xrightarrow{L^p_1} g_{\beta\alpha} \Rightarrow g_{i,\beta\alpha} \cdot a_{i\beta} - a_{i\alpha} g_{i,\beta\alpha}$ converge strongly in L^p_1

$\Rightarrow dg_{i,\beta\alpha} \xrightarrow{L^p_2} \Rightarrow$