$$0=\int_{B}\langle d_A\xi, 0\rangle=\int_{\partial B} d(\xi * a) \pm \int_B \langle \xi, d_A^* a\rangle \quad \forall \xi \Rightarrow *a|_{\partial R}=0,\ d_A^* a=0.$$

<u>Uhlenbeck's Fundamental Lemma</u> Let $P=B^n\times G\to B$, $n\leq 4$.

$\exists\, c, a$ s.t. $A\in\mathcal{A}_{L^2_1}$ if $\int_B |F_A|^2<\varepsilon \Rightarrow \exists u\in \mathcal{G}_{L^2_2}$ s.t. $u\cdot A=\Gamma+a$ where Γ is the trivial connection. Then a) $\int_B |\nabla_\Gamma a|^2+|a|^2\leq C\int_B |F_A|^2$,

b) $d_\Gamma^* a=0$ and $*a|_{\partial B^n}=0$.

Remark: b) is the "slice" condition b) $\Leftrightarrow \int_B \langle d_\Gamma \xi, a\rangle=0 \ \forall \xi\in\Omega^0(B,\mathrm{ad}P)$

$$d\, \cancel{-}tr(\xi * a) = \cancel{-}tr(d_A\xi\wedge * a) = tr(\xi\wedge d_A * a)$$

$$\Downarrow$$

$$\int_{\partial B} -tr(\xi * a) = -\int_B tr(d_A\xi\wedge * a) = \int_B tr(\xi\wedge *(*d_A *)a.$$

If $A\rightsquigarrow \Gamma+a$, $d_A^* a=0$, $*a|_{\partial B}=0$ and $\|a\|_{L^2_1}$ is small, estimate

a) follows. $\|F_A\|_{L^2(B)}=\|F_{\Gamma+a}\|_{L^2(B)}$, $F_{\Gamma+a}=da+a\wedge a$. $\int |da+a\wedge a|^2 \geq \int_B |da|^2 - c\int_B |a|^4$

Since $n\leq 4$, $L^2_1\hookrightarrow L^4$, $\int |a|^4\leq c_1\|a\|^4_{L^2_1}$

If $\int |da|^2 \geq c_2 \|a\|_{L^2_1}$ (*)

$\Rightarrow \int |F_A|^2 \geq c_2 \|a\|_{L^2_1}^2 - cc_1 \|a\|_{L^2_1}^4$

$\geq (c_2 - cc_1 \|a\|_{L^2_1}^2) \|a\|_{L^2_1}^2$. When does (*) hold?

We have $d^*a = 0$, $i_n^*(a) = 0$, $\int_B |da|^2 = \int_B |(d+d^*)a|^2 \geq \|a\|_{L^2_1}^2$.

Thm: $H^1(M)$ (M mfld w/ ∂) is represented harmonic form $\{\theta \mid (d+d^*)\theta = 0$ and $*\theta|_{\partial M} = 0\}$. Find θ by minimizing $\|\theta + df\|_{L^2}^2$

$d\theta = 0$

$d+d^* : \{\alpha \mid *\alpha|_{\partial M} = 0\} \to \Omega^0_{L^2} + \Omega^2_{L^2}$ is injective if $H^1(M) = 0$.

$(d+d^*)^2 = \nabla^*\nabla + \underbrace{\text{Riemannian curvature action}}_{0 \text{ in flat ball}}$

$\Downarrow$

$\int |(d+d^*)a|^2 = \int |\nabla a|^2$ This equality holds on the B^4 with $*a|_{\partial B} = 0$.

(In general ~~the~~ when there are singular spaces of flat connections, norm doesn't estimate distance)

Problems: (1) There might be a small energy connection ~~not gauge eq~~ without gauge representative near trivial connection.

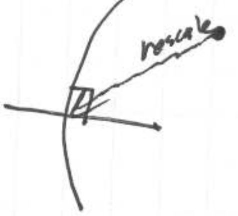
(2) Assumed the connection was gauge-equivalent into the slice.

We showed that if we fix $\Gamma \in \mathcal{A}_{L^2_k}$, then $\exists\, \varepsilon$ s.t. $\tilde{S}_{\Gamma,\varepsilon} = \{\Gamma + a \mid d_A^* a = 0,\ *a|_{\partial M},\ \|a\|_{L^n} < \varepsilon\}$ is injective into the quotient space $\mathcal{A}/\mathcal{G}$

$\tilde{S}_{\Gamma,\varepsilon} \times \mathcal{G}/\mathrm{Stab}_\Gamma \to \mathcal{A}$ is an injection.

Hope: If A is an L^n-connection, $\exists\, u \in \mathcal{G}_{L^n_1}$ so that $u^*A = \Gamma + a$, $a \in \tilde{S}_{\Gamma,\varepsilon}$.

Lemma: $\exists$ a $\mathcal{G}$-invariant nbhd $\mathcal{U}$ of Γ in the L^n topology so that every connection $A \in \mathcal{U}$ is gauge-equivalent into the slice $\tilde{S}_{\Gamma,\varepsilon}$.



rescaling decreases YM energy.

Want to prove for L^n ball of radius δ. Can't use implicit function thm since we are on borderline. Instead, use open/closed argument.

Want to sharpen result to show $m: (\tilde{S}_{\Gamma,\varepsilon} \times \mathcal{G}_{L^2_{k+1}})/\mathrm{Stab}(\Gamma) \to \mathcal{A}_{L^2_k}$ is a diffeo onto its image. Need to show differential is an iso. This map is equivariant, so enough to check at $(\Gamma+a, 1)$.

$$D_{(\Gamma+a,1)}\, m(\alpha,\xi) = d_\Gamma \xi + [a,\xi] + \alpha.$$

Suppose for simplicity that $\ker(d_\Gamma) = 0$. Prove injectivity.
Suppose $D_{(\Gamma+a,1)}\, m(\alpha,\xi) = 0$. Take inner product with $d_\Gamma \xi$

$$\Rightarrow \int |d_\Gamma \xi|^2 + \int \langle [a,\xi], d_\Gamma \xi\rangle + \int \langle \alpha, d_\Gamma \xi \rangle = 0$$

$\int \langle \alpha, d_\Gamma \xi \rangle \to 0$ since α is in slice

$\frac{1}{n} + \frac{n-2}{2n} + \frac{1}{2} = 1$

$$\Longrightarrow \int |d_\Gamma \xi|^2 \le \|a\|_{L^n} \|\xi\|_{L^{2n/n-2}} \|d_\Gamma \xi\|_{L^2}. \qquad L^2_1 \hookrightarrow L^{2n/n-2}$$

$$\le C \|a\|_{L^n} \|d_\Gamma \xi\|^2_{L^2}$$

(*) $\|\xi\|_{L^2_1} \le C \|d_\Gamma \xi\|_{L^2}$ (since $\ker(d_\Gamma) = 0$).

$$\Rightarrow \int |d_\Gamma \xi|^2 = 0 \text{ if } \|a\|_{L^n} \text{ small.}$$

$\boxed{(n \neq 2)}$

If $\ker d_\Gamma \neq 0$, we can assume that (α,ξ) is a representation of the tangent space of the quotient $(\tilde{S}_{\Gamma,\varepsilon} \times \mathcal{G})/\mathrm{Stab}(\Gamma)$ at $(\Gamma+a, 1)$.
(α,ξ) is orthogonal to $([a,\xi_0],\xi_0)$, $\xi_0 \in \ker d_\Gamma \Rightarrow \xi \perp \ker d_\Gamma$. For such ξ, estimate (*) holds so the argument goes through.

Thus $\xi = 0 \Rightarrow \alpha = 0 \Rightarrow \mathcal{D}_{(\Gamma+a,1)}{}^{m}$ is injective.

$\mathcal{D}_{(\Gamma+a,1)}{}^{m}$ is a continuously varying family of Fredholm operators.

$\mathcal{D}_{(\Gamma,1)}{}^{m}$ is invertible $\Rightarrow$ ind $=0$, so injectivity $\Rightarrow$ isomorphism.

$\mathcal{D}_{(\Gamma+a,1)}{}^{m} : L^2_k \times L^2_{k+1} \to L^2_k \quad k \gg 1.$

On L^2_k ball, m maps diffeomorphically to its image.

WTS image contains a L^n-ball,