$N^0(2,1) = \{ F_A = \begin{bmatrix} -1/2 & \\ & -1/2 \end{bmatrix}\omega \mid \det(A) = \delta \} / \mathcal{G}_2.$ 11/03/08 Moduli spaces

$\mathcal{A}_\delta = \{ A \ L^2_1\text{-connection in } E \mid \det A = \delta \}.$

$\mathcal{F}: \mathcal{A}_\delta \to \Omega^2(\Sigma, su(E)),\ \mathcal{F}(A) = F_A.$

Observation: If A is irreducible $(\mathrm{Stab}(A) = \{\pm 1\}) \Rightarrow d_A\mathcal{F}$ is surjective

$\Rightarrow \mathcal{F}^{-1}(\begin{bmatrix} -1/2 & \\ & -1/2\end{bmatrix}\omega)$ is smooth, $\mathcal{G}_\delta$ acts freely $\Rightarrow N^0(2,1)$ is a mfld.

$T_{[A]} N^0(2,1) = \ker(\mathcal{D}_A\mathcal{F}) / \mathrm{Im}(d_A : \Omega^0(\Sigma; su(E)) \to \Omega^1(\Sigma, su(E))$

$$\underset{T_I\mathcal{G}_\delta}{\shortparallel} \qquad \underset{T_A\mathcal{A}_\delta}{\shortparallel}$$

$$0 \to \Omega^0(\Sigma, su(E)) \xrightarrow{d_A} \Omega^1(\Sigma, su(E)) \xrightarrow{d_A} \Omega^2(\Sigma, su(E)) \to 0. \quad (*)$$

If A is YM $\Rightarrow (*)$ is a complex, $d_A^2 = [F_A, \bullet] = 0$ since F_A is central.

$\mathrm{Ind} = \dim N^0(2,1)$. If E is trivial (or more generally if $su(E)$ trivial), we compute the index by htpy invariance, take connection to be trivial $\Rightarrow$

$0 \to \Omega^0(\Sigma) \otimes su_2 \xrightarrow{d} \Omega^1(\Sigma) \otimes su_2 \xrightarrow{d} \Omega^2(\Sigma) \otimes su_2 \to 0 \Rightarrow \mathrm{ind} = 3(2-2g) \Rightarrow \dim = 6g-6$

If we use G connections, we get $\dim(G)(2g-6)$ as the dimension.

when is $su(E)$ trivial? $\langle c_1(E), [\Sigma] \rangle = 1$. Take $E = L \oplus \mathbb{C}$

$\langle c_1(L), [\Sigma] \rangle = 1$. $su(E) \cong \mathbb{R} \oplus L$. $w_2(su(E)) = w_2(L) = c_1(L) \bmod 2$.

$su(E) \oplus su(E)$ is trivial since $w_2(\cdots) = 0$. Thus

$2\,\mathrm{ind}(A) = \mathrm{ind}\ 0 \to \Omega^0(\Sigma, su(E) \oplus su(E)) \to \Omega^1 \to \cdots) \Rightarrow 2\,\mathrm{ind}(*) = 2(6-6g)$

$\Rightarrow \mathrm{ind}\,* = 6-6g$ in general.

$g=1$: $\dim N^0(2,1)=0$. Look for rep $\pi_1(T^2)\to SO_3$ so that associated $\mathbb{R}^3$-bundle is nontrivial.

$\lambda_1, \lambda_2, \lambda_3 \to T^2$ are nontrivial line bundles, non isomorphic.

$\lambda_1 \oplus \lambda_2 \oplus \lambda_3 \to T^2$, $w_1(\lambda_1+\lambda_2+\lambda_3) = w_1(\lambda_1)+w_1(\lambda_2)+w_1(\lambda_3)=0$.
(Bundle with discrete holonomy is necessarily flat) $w(\lambda_1+\lambda_2+\lambda_3) = (1+x_1)(1+x_2)(1+x_3) = 1+x_1+x_2+x_3 + x_1x_2+x_2x_3+x_3x_1 =$ gen of $H^2(T^2;\mathbb{Z}_2)$

<u>Alternative:</u> $SU_2 = Sp_1 =$ Unit quaternions. A, B generators of $\mathbb{Z}^2$.

$A, B \to SU_2$, $[A,B]\in Z(SU_2) = \pm\mathbb{1}$, $A\mapsto I$, $B\mapsto J$, $ABA^{-1}B^{-1} = K^2 = \mathbb{1}$.

$$A\mapsto \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1\end{bmatrix} \quad B\mapsto \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1\end{bmatrix}.$$

$N^0(2,1)$ is an example of a Marsden-Weinstein quotient.
(M,ω) a symplectic mfld, G action on M preserving ω, can construct a moment map $\mu: M\to \mathfrak{g}^*$ so that X a v.f. on (M,ω), $\mathcal{L}_X\omega = 0 \Rightarrow$
$d i_X\omega + i_X \underbrace{d\omega}_{0} = 0 \Rightarrow d(i_X\omega)=0$. $dH \overset{?}{=} i_X\omega$. $\mu \Leftrightarrow \mathfrak{g}\to C^\infty(M)$,
(choice of Hamiltonian for each $\xi\in\mathfrak{g}$.) $\mu(x\cdot g) = Ad(g^{-1})^*\mu(x)$.

$\mu^{-1}(0)/G$ is a symplectic mfld if 1) 0 is a regular value of μ,

and 2) $Stab(x)\in\mu^{-1}(0)$ is constant in G.

$\mathcal{A}_\delta$ is symplectic since $T_A\mathcal{A}_\delta = \Omega^1(\Sigma;\mathfrak{gu}(E))$, $\omega(a,b) = \int_\Sigma tr(a\wedge b)$.

$\omega_A(a,b) = -\omega(b,a)$. ω is closed since it's constant.

(Def of Closed: Restrict to any f.d. subspace and get something closed. $\omega|_S$ is closed for all finite dim submflds)

Def. of nondegenerate: There is a natural Riemannian metric on $\mathcal{A}_\delta$
$\omega_A(a,b) = \langle a, *b\rangle_A$, $\langle a,b\rangle_A = \int -tr(a\wedge *b)$, $\langle a,a\rangle_A = 0 \Rightarrow a=0$.

$\mathcal{G}$ acts on $\mathcal{A}_\delta$. Claim $\mathcal{G}$ preserves ω, $\omega_{g\cdot A}(g\cdot a, g\cdot b) \overset{?}{=} \omega_A(a,b)$

$$\int tr(ga\wedge gb) = tr(gag^{-1}\wedge gbg^{-1}) = \int tr(a\wedge b) \checkmark.$$

$\mu: \mathcal{A}_\delta \to (T_1\mathcal{G})^*$ is the curvature. $T_{\mathbb{1}}\mathcal{G} = \Omega^0(\Sigma;\mathfrak{gu}(E))$, $T_1\mathcal{G}^* = \Omega^2(\Sigma;\mathfrak{gu}(E))$.

$\mu : M \to \mathfrak{g}^*$, $\xi \in \mathfrak{g}$. Check that $d\langle \mu(x), \xi\rangle = \pm i_\xi \omega \Leftrightarrow X\langle \mu(x), \xi\rangle$

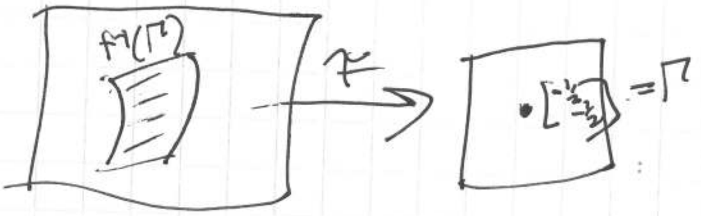
$= \omega(\xi, X)$.

$A \mapsto \int_\Sigma tr(F_A \wedge \xi)$

$A + ta \rightsquigarrow \frac{d}{dt} tr(F_{A+ta} \wedge \xi) = \int tr(d_A a \wedge \xi) = \pm \int tr(a \wedge d_A \xi)$

Formally we should see that $N^0(2,1)$ is a symplectic mfld.

$\mathcal{F}^{-1}\left(\begin{bmatrix} -1/2 & \\ & -1/2 \end{bmatrix} *_\Sigma 1\right)$



Relation w/ Geometric Invariant Theory

(M, ω, J) a Kähler mfld on which G compact Lie group acts, preserving ω and J. Often can extend action of G to $G^{\mathbb{C}}$ complex Lie group containing G as a maximal compact subgroup, Eg.

$G = U(n)$, $G^{\mathbb{C}} = Gl(n, \mathbb{C})$, $G = SU(n)$, $G^{\mathbb{C}} = Sl(n, \mathbb{C})$.

$\mathfrak{g} \to Vect(M)$, Want extension $\mathfrak{g} \otimes \mathbb{C} \to Vect(M)$. Take $i\xi \mapsto J\hat{\xi}$.

Must check to see if this ~~integrates~~ exponentiates to the group level.

An algebraic geometer would try $M/G^{\mathbb{C}}$, but this is often not even Hausdorff.

$M^s \subset M$ is defined to be the union of all closed orbits of $G^{\mathbb{C}}$. $M^{\mathbb{C}}/G^{\mathbb{C}}$ has compactifications which are varieties or even Kähler, denoted $M/\!/G^{\mathbb{C}}$. Kempf-Ness (in f.d. situation), $M/\!/G^{\mathbb{C}} = \mu^{-1}(0)/G$.

$G_1 := \{u \in \Gamma(\Sigma, End(E)) \mid \det u = 1,\ u^* u = 1\}$.

$G_1^c := \{u \in \Gamma(\Sigma, End(E)) \mid \det u = 1\}$.

Choose a metric on $\Sigma \Rightarrow *: \Omega^1(\Sigma) \circlearrowleft$, $*^2 = -1 \Rightarrow \Omega^1(\Sigma) \otimes \mathbb{C} = \Omega^{1,0}(\Sigma) \oplus \Omega^{0,1}(\Sigma)$.

$E \to \Sigma$ hermitian vector bundle. Choose a connection A in $E \Rightarrow d_A \to \Omega^1(\Sigma, E)$, $\downarrow \pi^{0,1}$, $\bar\partial_A : \Omega^{0,0}(\Sigma; E) \to \Omega^{0,1}(\Sigma; E)$

If A is unitary, then $\bar\partial_A s = \partial_A \bar{s}$.

$\mathcal{G}^c$ acts on $\bar\partial_E$ operators in E by $\bar\partial_E : \Omega^{0,0}(\Sigma, E) \to \Omega^{0,1}(\Sigma, E)$,

$\bar\partial_E(fs) = (\bar\partial f)s + f\bar\partial_E s$. $g \in \mathcal{G}^c$, $\bar\partial_E^g = g^{-1}\bar\partial_E g$. Let $\mathcal{E}_\delta = \bar\partial$-ops in E

$\swarrow \mathcal{G}^c$ acts here

$\mathcal{A}_\delta \to \mathcal{E}_\delta \to \mathcal{A}_\delta$

$A \mapsto \bar\partial_A$

$$\bar\partial_E \mapsto (s \mapsto \bar\partial_E s + \overline{(\bar\partial_E s)})$$

We need to be able to identify which $\bar\partial$-ops have closed orbit: "stable vector bundles".

If Kemf-Ness holds $\Rightarrow$ Stable v-b$/\mathcal{G}^c$ = minimal YM connections$/\mathcal{G}$,