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In the Morse-Bott situation, we have $dist(x, C) \le |\nabla_x f|$ $\forall x$ in some nbhd of C. $\Rightarrow$ If $\gamma(t)$ is a flow line with $\gamma(t)$ in this nbhd $\forall t>0$ and $f(\gamma(t)) > f(C)$ ($f(C)=0$, say.) $\forall t>0$ $\Rightarrow f(\gamma(t)) \le Ce^{-\alpha t}$, $C, \alpha > 0$.

$$\frac{d}{dt} f(\gamma(t)) = -|\nabla_{\gamma(t)} f|^2 \le -\alpha f(\gamma(t)), \quad \alpha > 0.$$

$$\frac{\frac{df}{dt}}{f} \le -\alpha \Rightarrow \ln f(\gamma(t)) - \ln f(\gamma(0)) \le -\alpha t$$

$f(\gamma(t)) \le f(\gamma(0)) e^{-\alpha t}$. $\quad |\gamma(t+1) - \gamma(t)| \le \int_t^{t+1} |\nabla_{\gamma(s)} f| \, ds$

$\le \left(\int_t^{t+1} |\nabla_{\gamma(s)} f|^2 ds\right)^{1/2} \le \left(f(\gamma(0)) e^{-\alpha t}\right)^{1/2} \Rightarrow |\gamma(t+u) - \gamma(t)| \le Ce^{-\frac{\alpha}{2} t}$ thn.

$\Rightarrow \lim_{t \to \infty} \gamma(t)$ exists and $\gamma(t)$ approaches limit exponentially fast.

$\int |\gamma|^2 e^{2\delta t} dt < \infty$ for δ small enough.

$$\int_0^\infty |\dot\gamma(t)|^2 e^{2\delta t} dt < \infty$$

$$\parallel$$

$$\int_0^\infty -\dot f e^{2\delta t} dt = f(0) + \int_0^\infty \frac{f e^{2\delta t}}{\delta} dt,$$ finite for δ small enough.

X_j Morse stratification.

when does this inject?

$$\begin{tikzcd}
H^{*-1}_G(X_{j-1}) \arrow[r, "\delta"] & H^*_G(X_j, X_{j-1}) \arrow[r] \arrow[d, phantom, "\cong"] & H^*_G(X_j) \arrow[d] \\
H^{*-1}_G(\partial W^u(C)) \arrow[r, "\delta"] & H^*_G(\overline{W^u(C)}, \partial W^u(C)) \arrow[r] \arrow[d, phantom, "\cong"] & H^*_G(W^u(C)) \arrow[d, phantom, "\cong"] \\
& H^{*-n}_G(C) \arrow[r, "\cup e(\nu_-)"] & H^*_G(C)
\end{tikzcd}$$

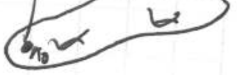
$\cup e(\nu_-)$ is injective. Need to see that

$N^\delta(2,1)$

$c_i = \{$crit. pts. (mod gauge) of YM compatible with a splitting $E = E_1 \oplus E_2$.

$\langle c_1(E_1) + c_1(E_2), [\Sigma] \rangle = 1$, $\langle c_1(E_1), [\Sigma] \rangle = i \ge 1$.

A connection A in E with $\det A = \delta$ in $\Delta = \det(E)$ can have stabilizer either U_1 or ± 1. The connections in c_i
have stabilizer U_1.

$C_i \times EG/G$  $G_0 = \{u \in G \mid u|_{x_0} = Id\}$. $1 \to G_0 \to G \to SU_2 \to 1$

G_0 acts freely on $\mathcal{A}$. $\mathcal{B}^0 = \mathcal{A}/G_0$. $H_G(\mathcal{A}) = H_{SU_2}(\mathcal{B}^0)$.

$$\cancel{\begin{array}{ccc} ? & \to & C_i/G_0 \\ \downarrow SU_2/U_1 \cong S^2 & & \downarrow \\ C_1 & \to & C_g/G \cong T^{2g} \end{array}} \qquad \cancel{\nu_- \simeq H^0(\Sigma; \Omega^{1,0} \otimes E_1 \times E_2^*)}$$

C_i = critical pts of YM in $\mathcal{A}$. $\begin{array}{c} \nu_- \\ \downarrow \pi \\ C_i \end{array}$ $\pi^{-1}(A) = \ker(\bar{\partial}_A : \Omega^{1,0}(\Sigma, E_1^* \otimes E_2) \to \Omega^{1,1}(\Sigma, E_1^* \otimes E_2))$.

Gauge group acts. If a connection has stabilizer, gauge group identifies points ~~others~~ in the same fiber.

G_0 acts freely on C_i (and ν_-). $\begin{array}{c} \nu_-^0 = \nu_-/G_0 \\ \downarrow_0 \\ C_i^0 = C_i/G_0. \end{array}$ SU_2 acts on everything.

Need to show $e_{SU_2}(\nu_-^0) \in H^*_{SU(2)}(C_i^0)$ is not a zero-divisor.

$U_1 \subset SU_2$ which acts trivially on C_i^0.

$$\begin{array}{ccc} C_i^0 \times_{U_1} ESU_2 & \xrightarrow{p} & C_i^0 \times_{SU_2} ESU_2 \\ \| & & \\ C_i^0 \times BU_1 & & \end{array}$$

$$H^*_{SU_2}(C_i^0) \xrightarrow{p^*} H^*(C_i^0 \times BU_1)$$

Exercise: The image of p^* is a summand.

$p^*(e(\nu_0^-)) \in H^*(C_i^0 \times BU_1)$. Not a zero-divisor if it has a nonzero component in $H^*(BU_1)$. $p^*\nu_0^-|_{[A] \times BU_1} = EU_1 \times_{U_1} \mathbb{C}^n \leftarrow H^0(\Sigma, \Omega^{1,0} \otimes E_1^* \otimes E_2)$

$\Rightarrow p^*(e(\nu_0^-))|_{BU_1}$ is nonzero $\Rightarrow p^*(e(\nu_-))$ is not a zero-divisor.

$2^n z^n$

$\begin{bmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{bmatrix}$ $E_1 \oplus E_2$

$\Rightarrow$ weight $e^{-2i\theta}$

$P_t(N^0(2,1)) = \sum t^k \dim(H^k_{\mathbb{Q}}(N^0(2,1)))$.

$H^*_G(\mathcal{A}) = \mathbb{Q}[1\ 2\text{-d},\ 2g\ 3\text{-d},\ 1\ 4\text{-d}] \Rightarrow P_t(H^*_G(\mathcal{A})) = \frac{1}{1-t^2}(1+t^3)^{2g}\frac{1}{1-t^4}$.

$P_t(H_G(\mathcal{A})) = \sum t^{n_-(i)} P_t(H_G(C_i)) + P_t(N^0(2,1))$.

$H_G(C_i) = H_{SU_2}(C_i^0)$. $\quad \begin{array}{c} S^2 \to C_i^0 \\ \downarrow \\ T^{2g} \end{array}$ $\quad$ s.s. collapses, so

$$\begin{array}{c} H^*_{SU_2}(C_i^0) \cong H^*_{SU_2}(S^2) \otimes H^*(T^{2g}) \\ \| \\ H^*_{U_1}(pt) = \text{polynomial algebra.} \end{array}$$

$P_t(H_G(C_i)) = \frac{1}{1-t^2}(1+t)^{2g}$.

$n_-(i) = 2(2i-1+g-1)$.

$$\Rightarrow P_t(N^0(2,1)) = \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)} - \sum_{i=1}^{\infty} t^{2(2i+g-2)} \frac{(1+t)^{2g}}{1-t^2}$$

$$= \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)} - \frac{(1+t)^{2g}}{1-t^2} t^{2g} \frac{1}{(1-t^4)} = \frac{(1+t^3)^{2g} - (1+t)^{2g} t^{2g}}{(1-t^2)(1-t^4)}.$$

If $P_t(N^0(2,1))$ is the Poincaré poly of a mfld, then

$$P_{1/t}(N^0(2,1)) = t^{-\dim(N^0(2,1))} P_t(N^0(2,1)).$$

$$\| $$

$$\frac{(1+t^{-3})^{2g} - (1+t^{-1})^{2g} t^{-2g}}{(1-t^{-2})(1-t^{-4})} = \frac{t^{-6g}\left((1+t^3)^{2g} - (1+t)^{2g} t^{2g}\right)}{t^{-6}(1-t^2)(1-t^4)} = t^{-(6g-6)} P_t(N^0(2,1)).$$

Why is $N^0(2,1)$ a manifold? $N^0(2,1) = \left\{A \mid *F_A = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\right\}/\mathcal{G}$.

$\mathcal{A}_{L^2_1} \xrightarrow{\mu} L^2(\Sigma; \mathfrak{su}(E))$ $\quad N^0(2,1) = \mu^{-1}\left(\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\right)/\mathcal{G}$.

$A \mapsto *F_A$

$\mathcal{F}(A) = dA + A \wedge A$ is a smooth map. $D_A \mathcal{F}(a) = *d_A a$.

$*d_A : \Omega^1(\Sigma; \mathfrak{su}(E)) \to \Omega^0(\Sigma; \mathfrak{su}(E))$. The L^2 complement of $\mathrm{Im}(*d_A)$ $= \ker(d_A : \Omega^0(\Sigma; \mathfrak{su}(E)) \to \Omega^1(\Sigma; \mathfrak{su}(E))) = \{$parallel sections of $\mathfrak{su}(E)\}$.

Since $A \in N^0(2,1)$ is by assumption irreducible, this is $\{0\}$.

"You could take this as a computation of its dimension, but it's a little bit sick."

$\Rightarrow D_A f$ is surjection, $T_{[A]} N^0(2,1) = \ker(d_A : \Omega^1 \to \Omega^2) / \mathrm{Im}(d_A : \Omega^0 \to \Omega^1)$.

$\Omega^0(\Sigma, \mathfrak{su}(E)) \xrightarrow{d_A} \Omega^1(\Sigma; \mathfrak{su}(E)) \xrightarrow{d_A} \Omega^2(\Sigma; \mathfrak{su}(E))$

has index 6 − 6g.