Def: $f: M \to \mathbb{R}$ is Morse-Bott if the crit. set. $C = \{x \mid d_x f = 0\}$ can be written as union of components $C = \bigcup C_\alpha$ and 1) Each C_α is a submfld of M, 2) $\ker(\mathrm{Hess}_x f) = T_x C_\alpha$ $\forall x \in C_\alpha$.

eg: 

2) $\Rightarrow$ if Υ is a transverse submfld to C at x and $\Upsilon \cap C = x$, then $f|\Upsilon$ has a nondeg. crit point at x.

Let ν_α = normal bundle of C_α, $\nu_\alpha = \nu_\alpha^+ \oplus \nu_\alpha^-$, $\mathrm{Hess}_\alpha(f)|\nu_\alpha^\pm$ is pos (neg) definite. For each C_α we can define

$W^+_{C_\alpha} = \{x \in M \mid \lim_{t\to\infty} \varphi_t(x) \in C_\alpha\}$, $W^-_{C_\alpha} = \{x \in M \mid \lim_{x \to -\infty} \varphi_t(x) \in C_\alpha\}$.

Thm: $W^\pm_{C_\alpha}$ are submflds of M, and $W_{C_\alpha}^\pm$ is diffeo to $\nu_\alpha^\pm$.

Let's see that $\forall x \in C_\alpha$ $W^\pm_x$ are submflds diffeo to $\mathbb{R}^{n_\pm}$, $n_\pm$ = dim fiber of $\nu^\pm$.

Work locally: $\mathbb{R}^n = \mathbb{R}^{n_+} \oplus \mathbb{R}^{n_-} \oplus \mathbb{R}^c$, $c = \dim(C_\alpha)$. f can be assumed to be of the form $f(x^+, x^-, y) = q_y(x^+, x^-)$, $y \mapsto q_y$ is a smooth map $\mathbb{R}^c \longrightarrow$ Quad form on $\mathbb{R}^{n_+} \times \mathbb{R}^{n_-}$ of signature (n_+, n_-).

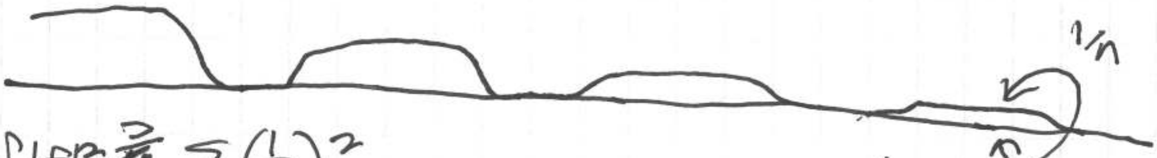
$H_1 = \{\gamma : [0,\infty) \to \mathbb{R}^n \mid \int |\dot\gamma|^2 + |\gamma|^2 < \infty\}$. $H_0 = \{\gamma : [0,\infty) \to \mathbb{R}^n \mid \int |\gamma|^2 < \infty\}$.

$\mathbb{F}(\gamma) = (\dot\gamma + \nabla_\gamma f)$ $\quad \tilde{\mathbb{F}} : H_1 \to H_+ \oplus \mathbb{R}^{n_+}$. (Was a diffeo in a nbhd of zero in $\mathbb{R}^c = \{0\}$ case).

$\mathbb{F} : H_1 \to H_0$.

$\gamma \mapsto \frac{d}{dt}\gamma$ Doesn't have closed range.

$H_1 \to H_0$



$\int |f|^2 \geq \sum \left(\frac{1}{n}\right)^2 \cdot n = \infty$.

$\int |\dot f|^2 \leq 2 \sum \left(\frac{1}{n}\right)^2 < \infty$.

$H_{1,\delta} = e^{-t\delta} H_1$, $H_{0,\delta} = e^{-t\delta} H_0$. $H_{1,\delta}$ = completion of $C_0^\infty([0,\infty), \mathbb{R}^n)$ in the norm $\|f\|^2_{1,\delta} = \int_0^\infty e^{2t\delta}(|\dot f|^2 + |f|^2)\,dt$. (Similarly for $H_{0,\delta}$). Weighted L^2-spaces

$\mathbb{F} : H_{1,\delta} \to H_{0,\delta}$. $D_0 \mathbb{F}(\xi) = \frac{\partial \xi}{\partial t} + \mathrm{Hess}_0 f(\xi)$. $H_{1,\delta} \overset{e^{t\delta}}{\rightleftarrows} H_1$, $H_{0,\delta} \xrightarrow{e^{t\delta}} H_0$.

$$\begin{tikzcd} H_{1,\delta} \arrow[r, "DoH"] \arrow[d, "e^{t\delta}"] & H_{0,\delta} \arrow[d, "e^{t\delta}"] \\ H_1 \arrow[r, dashed] & H_0 \end{tikzcd}$$

$-\delta + \mathrm{Hess}_0 f$ is invertible if $\delta \notin \mathrm{Spec}(\mathrm{Hess}_0 f)$ if $0 < \delta < 1^{st}$ positive eigenvalue.

$\varepsilon \mapsto \frac{\partial}{\partial t}\varepsilon - \delta\varepsilon + \mathrm{Hess}_0(f)\varepsilon$.

$\tilde{F}: H_{1,\delta} \to H_{0,\delta} \oplus \mathbb{R}^{n_+}$. This has invertible differential at $\gamma \equiv 0$, so inverse function holds and $F^{-1}(0, x^+)$ parameterizes a collection of paths solving $\dot{\gamma} + \nabla_\gamma f = 0$ which converge to zero in forward time.

If $x_1 \in W_x^s$, and x_1 is sufficiently close to x, then $\int_0^t |\dot{\gamma}|^2 = f(\gamma(0)) - f(\gamma(t))$ $\Rightarrow \gamma(t) = \varphi_t(x_1) \in H_1$ and has small H_1 norm $\Rightarrow$ captured by IFT.

The Morse Bott case is a little more subtle and left to your imagination.

Proxy for distance to C.

<u>Assumed:</u> Suppose $\gamma(t)$ is a flow line. We want to see $\lim_{t \to \infty} \gamma(t)$ exists, and $\mathrm{dist}(\gamma(t), C) \to 0$ exponentially fast.

<u>Lemma:</u> $\exists$ nbhd $\mathcal{U}$ of C s.t. $|\nabla_x f| \geq \mathrm{dist}(x, C)$. (Equals Hessian plus lower-order stuff which doesn't destroy limit. <u>Deferred to next time.</u>

<u>The Morse Stratification of M:</u>

<u>Def:</u> A <u>stratification</u> (by manifolds) of a space X is a decomposition $M = \coprod_i S_i$ so <u>that</u> 1) S_i are manifolds (locally closed in X, i.e. $\forall x \in S_i$ $\exists$ $U \subset X$ open so that $\overline{S_i \cap U} = S_i \cap U$. 2) $\overline{S_i} \setminus S_i \subset \bigcup_{j<i} S_j$. Stratification $\Rightarrow$ Spectral sequence to compute homology of X. In some cases (~~cell decomposition~~ CW-cx) a complex.

Given a Morse fn (or Morse-Bott fn), S_0 = union of minima of f, S_i = union of W^u_{Cx} so that $\overline{S_i} - S_i \subset \bigcup_{j<i} S_j$. $X_i = \bigcup_{j \leq i} S_j$.

Understand how $H^*(X_i)$ changes as i grows.

$$H^*(X_i) \to H^*(X_i, X_{i-1}) \xrightarrow{\delta} H^{*+1}(X_{i-1})$$

$$\begin{array}{c} \cong \\ H^*(\overline{S_i}, \overline{S_i} - S_i) \xrightarrow{\delta} H^{*+1}(\overline{S_i} - S_i) \\ \| \\ \oplus \mathbb{Z} \end{array}$$

In the Morse-Bott situation, we need to understand if a sphere in W_x^u bounds a cycle in X_{i-1}. In the Morse-Bott situation, $H^*(X_i, X_{i-1}) \simeq H^*(V_-, V_- - S)$

$\cong$

$H^*(\mathrm{Thom}(V_-))$.

$$H^*(X_i, X_{i-1}) \xrightarrow{\delta} H^{*+1}(X_{i-1})$$
$$\wr\wr \qquad\qquad\qquad \downarrow$$
$$H(\nu_-) \to H^*(\nu_-, \nu_- - 0) \longrightarrow H^{*+1}(\nu_- - 0)$$
$$H^*(C_\alpha) \xrightarrow{\cup e(\nu_-)} H^{*+k}(C_\alpha) \xrightarrow{\delta} H^{*+1}(S(\nu_-))$$

We can guarantee that a class $\alpha \in H^*(\nu_+, \nu_- - 0)$ extends to a class $\tilde{\alpha} \in H^*(X_i)$ if $\alpha = \Phi^{-1}(e(\nu_-) \cup \beta)$ using Φ the Thom isomorphism. In the finite dimension case, not all classes can have this form. But in the infinite dimensional or equivariant case, this can happen for all classes.

In the equivariant case, we'd like to check $e(\nu_-) \in H^*_G(C_\alpha)$. If class is not a zero divisor, then $H^*_G(C_\alpha) \hookrightarrow H^*_G(M)$. (This happens in YM).