Morse Thy (M cpt mfld) $f: M \to \mathbb{R}$

Def: f is a Morse Function if all crit. pts. are nondeg, i.e. $\mathrm{Hess}_x f = \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)$ invertible.

3 viewpoints:

1) Morse - Use f to understand homology of M (or to show existence of crit pts from homology)
2) Bott - " " " htpy type of M (e.g. periodicity)
3) Smale - Diffeo type of M

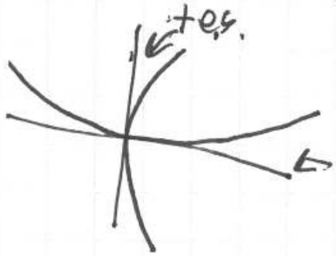
Choose a Riemannian metric g on M, $\dot{x} = -\nabla_x f$ downward gradient flow of f $\Rightarrow$ φ_t induced flow. If c is a crit. pt, then $W_c^{\pm} = \{x \in M \mid \lim_{t \to \pm\infty} \varphi_t(x) = c\}$.

Thm: (Hartman-Grobman (1959) local, Smale (1962) global)

$W_c^{\pm}$ is 1) a smooth mfld, 2) $\dim W_c^+$ (resp. W_c^-) $=$ # of pos (resp. neg) eigenvalues of $\mathrm{Hess}_c f$.
3) $W_c^{\pm}$ are diffeo $\mathbb{R}^{\mu_\pm}$, $\mu_\pm =$ # of $\pm$ eigenvalues.

Two methods of proof:

Hadamard (Graph transform) sketch



Set up the space of graphical submflds of a nbhd of c which pass through c and are tangent to -es.

graph of some function from minus eigenspace to positive eigenspace

$\|$
$\mathcal{G}$ The flow $\Phi_t \circlearrowright \mathcal{G}$.

Show that on some nbhd, this is a contraction mapping.
Have to pick some Banach space, and then send # of derivatives to infinity. Hopefully the region of contractibility doesn't shrink to zero. (Need to check this.)

Method 2 (Perron) The local W_c^+ is a set of initial conditions for $\dot{x} = \nabla_x f$ to have small solution $\forall$ positive time.
Find all $\gamma : [0,\infty) \to M$ so that $\gamma(t) \in$ nbhd of c $\forall t \in [0,\infty)$, and $\lim_{t\to\infty} \dot{\gamma}(t) = c$. $W_c^{+loc} = \{\gamma(0) \mid \gamma \text{ as above}\}$.

$$H_1 = \{\gamma : [0,\infty) \to \mathbb{R}^n \mid \int_0^\infty |\dot{\gamma}|^2 + |\gamma|^2 < \infty\}. \quad \mathbb{F} : H_1 \to H_0 = \{\gamma : [0,\infty) \to \mathbb{R}^n \text{ s.t. } \int |\gamma|^2 < \infty\},$$

$$\mathbb{F}(\gamma) = \dot{\gamma} + \nabla_\gamma f, \quad \mathbb{F}^{-1}(0) \text{ is a set of flow lines of } -\nabla f.$$

Prop: $H_1 \subset C^{1/2}$, and $\forall \gamma \in H_1$, $\lim_{t\to\infty} \gamma(t) = 0$.
Proof: $\gamma(t_2) - \gamma(t_1) = \int_{t_1}^{t_2} \dot{\gamma}(s)\,ds \Rightarrow |\gamma(t_2) - \gamma(t_1)| \le \int_{t_1}^{t_2} |\dot{\gamma}(s)|\,ds \le \left(\int_{t_1}^{t_2} |\dot{\gamma}(s)|^2 ds\right)^{1/2} (t_2 - t_1)^{1/2}$

Easy to see that $\lim_{t\to\infty} \gamma(t) = 0$.

$$\int_0^\infty |\nabla_\gamma f|^2 \le \int_0^\infty C|\gamma|^2 < \infty. \quad \text{since } (|\nabla_x f| < O(|x|) \text{ on some } B \text{ a ball about zero} \in \mathbb{R}^n)$$

Holds for $\{\gamma \in H_1 \mid \gamma(t) \in B \; \forall t\} = \mathcal{U}$. $|\nabla_x f - \nabla_y f| \le C|x-y|$.

$\mathbb{F} : \mathcal{U} \to H_0$. Claim: $\mathbb{F}$ is cts. $\|\mathbb{F}(\gamma_2) - \mathbb{F}(\gamma_1)\|^2 \le \int |\dot{\gamma}_1 - \dot{\gamma}_2|^2$
$+ \int |\nabla_{\gamma_1} f - \nabla_{\gamma_2} f|^2 \le \int |\dot{\gamma}_1 - \dot{\gamma}_1|^2 + \int c|\gamma_1 - \gamma_2|^2$ after possibly shrinking $\mathcal{U}$.
(But if we have to shrink, can just flow out to compensate since flow is smooth.)
Can check that $\mathbb{F}$ is in C^k $\forall k$, but possibly shrinking its domain each time.

$$D_0\mathbb{F} : H_1 \to H_0. \quad \mathbb{F}(\gamma) = \dot{\gamma} + \nabla_\gamma f, \quad \gamma = t\varepsilon$$

$$D_0\mathbb{F}(\varepsilon) = \dot{\varepsilon} + \mathrm{Hess}_0 f(\varepsilon) = \dot{\varepsilon} + H\varepsilon, \quad H : \mathbb{R}^n \to \mathbb{R}^n \text{ symmetric matrix.}$$

Prop: $D_0 F$ is ~~surjective~~ ~~injective~~ surjective with f.d. kernel.

$$\|D_0 F(\varepsilon)\|^2 = \int_0^\infty |\dot\varepsilon + H\varepsilon|^2 = \int_0^\infty \Big(|\dot\varepsilon|^2 + |H\varepsilon|^2 + 2\overset{\partial_t\langle \varepsilon, H\varepsilon\rangle}{\overset{\parallel}{\langle \dot\varepsilon, H\varepsilon\rangle}}\Big)\, dt$$

$$= \int_0^\infty \underbrace{\big(|\dot\varepsilon|^2 + |H\varepsilon|^2\big)}_{\geq c|\varepsilon|^2} dt + \langle \varepsilon(0), H\varepsilon(0)\rangle$$

$H = H_+ \oplus H_-$ $\quad \|D_\varepsilon F(0)\|^2 \geq \int_0^\infty |\dot\varepsilon|^2 + c|\varepsilon|^2 + \overset{\text{negative}}{\langle \varepsilon(0), H_- \varepsilon(0)\rangle}.$

$$\|D_\varepsilon F(0)\| - \langle \varepsilon(0), H_- \varepsilon(0)\rangle \geq c\|\varepsilon\|^2_{H_1}.$$

Thus the operator $\varepsilon \to (D_0 F(a), \Pi_-(\varepsilon(0)))$

$H_1 \to H_0 \oplus S_-$ is injective. $\quad (\mathbb{R}^n = S_+ \oplus \overset{\text{span of e.v.'s}}{S_-})$

Surjectivity: $\dot\varepsilon + H\varepsilon = \delta$, $\delta \in L^2$.

Know already from the estimate that $D_0 F$ has closed range, so enough to show dense range. Need to see that solution to $\dot\varepsilon + H\varepsilon = \delta$, $\delta \in C_0^\infty([0,\infty))$ have $\varepsilon \in H_1$. (ODE). The previous estimate $\Rightarrow \varepsilon \in H_1$.

$\dot\varepsilon + \lambda\varepsilon = \delta$ $\quad \lambda > 0$ $\quad \varepsilon(t) = \int_0^t e^{-\lambda(t-s)}\delta(s)\,ds$

exponential decay

$\varepsilon(t) = -\int_t^\infty e^{-\lambda(t-s)}\delta(s)\,ds$ $\quad \lambda < 0$

cptly supported if δ is

Thm: $\varepsilon \mapsto (F(\varepsilon), \Pi_+(\varepsilon(0)))$

$\hat F: \mathcal{U} \to H_0 \oplus S_+$ has invertible differential $\Rightarrow \exists V \subset \mathcal{U}$

so that $\hat F: V \to H_0 \oplus S_+$ is a diffeo onto its image.

If γ is a flow line, $\int_{t_1}^{t_2} |\dot\gamma|^2 = -\int_{t_1}^{t_2} \langle \dot\gamma, \nabla_\gamma f\rangle = -\int_{t_1}^{t_2} \frac{d}{dt} f(\gamma(t))\, dt = f(t_1) - f(t_2)$

The stable man. $\hat F^{-1}: W \subset H_0 \oplus S_+ \to V$, $\quad \hat F^{-1}|_{W \cap (\{0\} \oplus S_+)} \to V \xrightarrow{\text{eval at zero}} \mathbb{R}^n$.