$$\mathbb{C}^2 \to E \quad \det(E) = \Delta, \quad \delta \text{ a connection in } \Delta.$$
$$\downarrow$$
$$\Sigma$$

$$\mathcal{A}_\delta = \{ A \in \mathcal{A}_E \mid \det(A) = \delta \}$$

$$\mathcal{E}(A) = \int_\Sigma |F_A|^2. \qquad \mathcal{G}^1 = \{ u : P \to U_2 \mid u(pg) = g u(p) g^{-1} \text{ and } \det(u) = 1 \}.$$

$N(2,1)$ = set of minima of $\mathcal{E}$ mod $\mathcal{G}^1_0$. $\mathcal{E}$ has higher critical pts, $E = E_1 \oplus E_2$. $\langle c_1(E_i), [\Sigma] \rangle = \lambda_i$. Need to compute the Morse index of the critical pts. $Q_A = d_A^* d_A + *[*F_A, \cdot\,]$ Hessian of $\mathcal{E}$. $\rightsquigarrow$ $H_A = d_A^* d_A + d_A d_A^* + *[*F_A, \cdot\,]$. Has same negative spectrum as Q_A. Consider a critical pt A with corresponding splitting $E = E_1 \oplus E_2$

$$*F_A = -i\lambda_1 \oplus -i\lambda_2.$$

$$\Omega^1(\Sigma, \mathfrak{su}(E)) = \Omega^1(\Sigma, \varepsilon_{\mathbb{R}}) \oplus \Omega^1(\Sigma, L)$$
$$= \Omega^1(\Sigma, \varepsilon_{\mathbb{R}}) \oplus \Omega^{1,0}(\Sigma, L) \oplus \Omega^{0,1}(\Sigma, L).$$

$*F_A$ acts also on $\underset{\substack{\uparrow \\ 0}}{i\mathbb{R}} \oplus \underset{\substack{\nwarrow \\ (-i)(\lambda_1 - \lambda_2)}}{E_1 \otimes E_2^*}$.

Riemannian metric induces decomposition $\Omega^1(\Sigma, L) \simeq \Omega^{1,0}(\Sigma, L) \oplus \Omega^{0,1}(\Sigma, L)$

$$\underset{*}{\circlearrowleft} \qquad \Omega^{1,0} : -i, \qquad \Omega^{0,1} : i$$

$*^2 = -1$. $*(dx + i\,dy) = -i(dx + i\,dy)$

H_A preserves the decomposition $\Omega^1(\Sigma, su(E)) = \Omega^1(\Sigma; \mathbb{R}) \oplus \Omega^{1,0}(\Sigma, L) \oplus \Omega^{0,1}(\Sigma, L)$

$\underbrace{\Delta_A}_{\text{zero eigenvalues can come from here}}$ $\quad \underbrace{\Delta_A - (\lambda_1 - \lambda_2)}_{\text{negative and zero}}$ $\quad \underbrace{\Delta_A + (\lambda_1 - \lambda_2)}_{\text{positive only}}$

Assume $\lambda_1 > \lambda_2$.

Need to understand $\mathrm{spec}(\Delta_A)$ on $\Omega^1(\Sigma)$, $\Omega^{1,0}(\Sigma, L)$, $\Omega^{0,1}(\Sigma, L)$.

On $\Omega^1(\Sigma; \mathbb{R})$, $\Delta_A = \Delta$, so no negative ev's, zero eigenspace = harmonic 1-forms

$$\begin{array}{ccc} \Omega^{1,0}(\Sigma, L) & \xrightarrow{\bar\partial_A} & \Omega^1(\Sigma, L) \\ \uparrow \partial_A & & \uparrow \partial_A \\ \Omega^{0,0}(\Sigma, L) & \xrightarrow{\bar\partial_A} & \Omega^{0,1}(\Sigma, L) \end{array}$$

$d_A = \partial_A + \bar\partial_A$, on $\Omega^{0,0}$: $\partial_A = \pi^{1,0} d_A = \frac{1}{2}(1 + i*) d_A$

$\bar\partial_A = \pi^{0,1} d_A = \frac{1}{2}(1 - i*) d_A$

$\bar\partial_A : \Omega^{1,0}(\Sigma, L) \to \Omega^{1,1}(\Sigma, L)$, $\bar\partial_A = d_A|_{\Omega^{1,0}(\Sigma, L)}$

$\partial_A : \Omega^{0,1} \to \Omega^{1,1}$, $\partial_A = d_A|_{\Omega^{0,1}(\Sigma, L)}$.

$\Delta_A = d_A d_A^* + d_A^* d_A$, $\quad \Box_A = \partial_A \partial_A^* + \partial_A^* \partial_A$, $\quad \bar\Box_A = \bar\partial_A \bar\partial_A^* + \bar\partial_A^* \bar\partial_A$ act on $\Omega^{p,q}(\Sigma, L)$.

$\mathrm{Spec}^+(T) = \{\lambda \in \mathrm{Spec}(T) \mid \lambda > 0\}$.

If $T = D^* D$, $T' = D D^*$ where $D : H_1 \to H_2$ is Fredholm.

$D^* D \psi = \mu^2 \psi \Rightarrow D\psi \neq 0$ (since $\|D\psi\|^2 = \langle \psi, D^* D \psi \rangle = \mu^2 \langle \psi, \psi \rangle$) is an eigenvector of $D D^*$ with same eigenvalue $(D D^*)(D\psi) = D(D^* D \psi) = D(\mu^2 \psi) = \mu^2 D\psi$.

Now $\Box_A$ and $\bar\Box_A$ only have a single term.

$$\Rightarrow \mathrm{Spec}^+\big(\underbrace{\Box_A}_{\partial_A \partial_A^*}|_{\Omega^{0,0}(\Sigma, L)}\big) = \mathrm{Spec}^+\big(\underbrace{\Box_A}_{\partial_A^* \partial_A}|_{\Omega^{1,0}(\Sigma, L)}\big).$$

$$\mathrm{Spec}^+\big(\underbrace{\bar\Box_A}_{\bar\partial_A \bar\partial_A^*}|_{\Omega^{0,0}(\Sigma, L)}\big) = \mathrm{Spec}^+\big(\underbrace{\bar\Box_A}_{\bar\partial_A^* \bar\partial_A}|_{\Omega^{0,1}(\Sigma, L)}\big).$$

Lemma: On $\Omega^{0,0}$, $\square_A = \frac{1}{2}(\Delta_A + i * F_A)$ On $\Omega^{1,0}$ or $\Omega^{0,1}$:

$\overline{\square}_A = \frac{1}{2}(\Delta_A - i * F_A)$. $\square_A = \overline{\square}_A = \frac{1}{2}\Delta_A$.

Proof: On $\Omega^{0,0}(\Sigma, L)$, $\square_A = \partial_A^* \partial_A =$

$\partial_A = \pi^{1,0} \circ d_A$

$\partial_A^* = d_A^* \circ \pi^{1,0}$

$= d_A^* \frac{1}{2}(1 + i*) d_A$

$= -\frac{1}{2} * d_A * d_A - \frac{i}{2} * d_A \overset{(-1)}{**} d_A$

$= \frac{1}{2}(\Delta_A + i * F_A)$. Similarly for $\overline{\square}_A$. $\overline{\partial}_A = d_A \circ \pi^{1,0}$

$\overline{\partial}_A^* = \pi^{1,0} \circ d_A^*$

On $\Omega^{1,0}(\Sigma, L)$, $\overline{\square}_A = \overline{\partial}_A^* \overline{\partial}_A \rightsquigarrow \Omega^{1,0}$ ↺

Not what we want

$= \frac{1}{2}(1 + i*) d_A^* d_A = \frac{1}{2}(1 + i*)(-* d_A * d_A) = \frac{1}{2}(d_A^* d_A + i d_A d_A) = \frac{1}{2}(d_A^* d_A - d_A * d_A *) = \frac{1}{2}(d_A^* d_A + d_A d_A^*)$

$\square_A = \partial_A \partial^*_A = -\pi^{1,0} d_A * d_A * \pi^{1,0} = -\frac{1}{2}(1 + i*) d_A * d_A * \frac{1}{2}(1 + i*)$

$= -\frac{1}{4} d_A * d_A * + \frac{1}{4} * d_A * d_A \overset{(-1)}{**} - \frac{i}{4} * d_A * d_A \overset{\to i}{*} - \frac{i}{4} d_A \overset{*(*i)}{*} d_A \overset{(-1)}{**}$

$= \frac{1}{4}(d_A^* d_A + d_A^* d_A + d_A d_A^* + d_A d_A^*) = \frac{1}{2}\Delta_A$.

Similarly, $\Omega^{0,1}$: $\overline{\square}_A = \frac{1}{2}\Delta_A$.

On $\Omega^{0,0}(\Sigma, L)$, $\square_A = \frac{1}{2}(\Delta_A + i(-i\sigma)) = \frac{1}{2}(\Delta_A + \sigma)$

$\overline{\square}_A = \frac{1}{2}(\Delta_A - \sigma)$

$\Rightarrow \square_A = \overline{\square}_A + \sigma \Rightarrow \operatorname{Spec}(\square_A) \geq \sigma \Rightarrow$ Miracle 1: $\boxed{\operatorname{Spec}^+(\Delta_A|_{\Omega^{1,0}(\Sigma, L)}) \geq 2\sigma.}$

$\Rightarrow$ Morse index $= \#$negative eigenvalues of $H_A = \Delta_A + *[*F_A, *]$

$= \dim \ker(\Delta_A|_{\Omega^{1,0}(\Sigma, L)}) \overset{\text{sheaf cohomology}}{\cong} H^0(\Sigma, \Omega^{1,0} \otimes L) \overset{\text{Serre duality}}{\cong} H^1(\Sigma, L^*)^*$.

Riemann-Roch: $\dim H^0(\Sigma, L^*) - \dim(H^1(\Sigma, L^*)) = -c_1(L) + 1 - g$.

$\lambda_1 - \lambda_2 > 0 \Rightarrow c_1(L) > 0 \Rightarrow c_1(L^*) < 0 \Rightarrow H^0(\Sigma, L^*) = 0$

$\Rightarrow \boxed{\dim(H^1(\Sigma, L^*)) = \sigma + g - 1}$ σ is odd positive.