$S_{A,\varepsilon} = \{A + a \mid d_A^* a = 0,\ \|a\|_{L^p_k} < \varepsilon\} \subset \mathcal{A}^p_k \quad p(k+1) > n$ 9/29

$\hat{S}_{A,\varepsilon} = \{A + a \mid d_A^* a = 0,\ \|a\|_{L^n} < \varepsilon\} \subset \mathcal{A}^p_k$. Can't prove injectivity via IFT, but can still do it.

Proving something for L^n gives us a range of borderline results.

$\mathcal{A}$	L^4	L^2_1
$\mathcal{G}$	L^4_1	L^2_2
	top Banach mfld	smooth Banach mfld

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~~Def~~ Yang-Mills: $\mathcal{E}(A) = \int_M |F_A|^2$

$P \leftarrow$ unitary frame bundle of E

$P \to M$, $E \to M$

Euler-Lagrange equations for $\mathcal{E}$: $d_A^* F_A = 0$. $d_A F_A = 0$ (Bianchi)

$d_A^* F_A = 0 \iff d_A(*F_A) = 0 \overset{n=2}{\iff} *F_A$ is a parallel endomorphism.

Obs: Eigenvalues of a parallel endomorphism are constant.

$\xi = *F_A$, $\xi^n \in \Gamma(\Sigma, \mathrm{End}(E))$, $\mathrm{tr}(\xi^n) \in C^\infty(\Sigma)$, $d\,\mathrm{tr}(\xi^n) = \mathrm{tr}\left(\sum_{i=0}^{n-1} \xi^i d_A \xi\, \xi^{n-i-1}\right)$

$= n\,\mathrm{tr}(d_A \xi \cdot \xi^{n-1}) = 0 \Rightarrow$ e.v. are constant $\Rightarrow E = \bigoplus_{i\lambda \in \mathrm{spec}(E)} E_\lambda$

$\xi|_{E_\lambda} = i\lambda$.

Fix a Hermitian same E_λ $= [\frac{i}{2\pi} \mathrm{tr} F_A] = c_1(E_\lambda)_{\mathbb{R}} \in H^2(\Sigma; \mathbb{R})$

$\xi = i\lambda = 1$. $F_A = i\lambda\, \mathrm{Id} + 1$. $\mathrm{Tr}(F_A) = i\lambda\, \mathrm{rk}(E_\lambda) * 1$.

$\langle c_1(E_\lambda), [\Sigma] \rangle = -\frac{\lambda}{2\pi} \mathrm{rk}(E_\lambda)\, \mathrm{vol}(\Sigma)$.

If A is a Y-M connection on

$\Rightarrow E = \bigoplus E_\lambda$, $\mathcal{E}(A) = \int -\mathrm{tr}(F_A \wedge *F_A) = \bigoplus_\lambda \int -\mathrm{tr}((i\lambda\, \mathrm{Id}_{E_\lambda} \omega)(i\lambda\, \mathrm{Id}_{E_\lambda}))$

$= \sum_\lambda \lambda^2 \mathrm{rk}(E_\lambda) = -(4\pi^2) \sum \frac{c_1^2(E_\lambda)}{\mathrm{rk}\, E_\lambda}$ (if $\mathrm{vol}(\Sigma) = 1$).

Morse thy for $\mathcal{E}$ relates to the algebraic topology of $\mathcal{A}$ and $\mathcal{G}$.

$\mathcal{A}/\mathcal{G}$ is almost a classifying space for $\mathcal{G}$.
- $\mathcal{A}$ is contractible
- $\mathcal{G}$ almost acts freely.

Want also to understand topology of $B\mathcal{G}$.

Review: G topological group, $EG \to BG$ principal bundle, where EG is contractible

$$\begin{array}{ccc} P & \dashrightarrow & EG \\ \downarrow & & \downarrow \\ X & \to & BG \end{array}$$

Thm (Atiyah-Bott): $P \to M$ principal bundle, $\mathcal{G}_P$ = gauge group of transformations.

$$B\mathcal{G}_P \simeq [M, BG]_P \leftarrow \text{component of } P.$$

Proof: $E\mathcal{G}_P \overset{?}{=} [P, EG]^G \leftarrow$ G-equivariant

Claim: $[P, EG]^G$ is contractible.

Aside: What are we talking about? Function spaces.
Weakly htpy equivalent as long as space of mappings is cts.
Construct a heat flow on this space of maps
Palais proves that $L^p_k(X,Y) \underset{\text{htpy}}{\simeq} C^0_{\substack{\text{compact}\\ \text{open}}}(X,Y)$, X, Y are C^∞ mflds.
"Global analysis"
$L^p_k(X,Y) \subset C^0(X,Y)$

$$\pi_0(L^n_1(X^n, Y)) = \pi_0(C^0_{co}(X,Y)).$$

Suppose: P = trivial, $P = M \times G$, $[P, EG]^G = [M, EG] \simeq *$

In general, cover M by balls (metric balls) $B(x_i, r_i)$ so that $B(x_i, r_i/2)$ still cover. On each ball, restriction is contractible.

On $B(x_i, r_i)$, $[P|_{B(x_i,r_i)}, EG]^G$ is contractible. r,

$\varphi : B(x_i, r_i) \to [0,1]$. $\varphi(B(x_i, r_i/2)) = 1$, $\varphi(\partial B(x_i, r_i)) = 0$.

$f : P \to EG$. $r_t f(p) = \begin{cases} f(p), & p \notin B(x_i, r_i), \\ r_{t\varphi(\pi(p))} f(p), & p \in B(x_i, r_i). \end{cases}$

Proceed to $B(x_2, r_2)$ and repeat.

$[M^n, BG] = \bigcup_{\substack{[P] \\ \text{iso types}}} [M, BG]_P$ = union into components.

$[S^n, BG]_* = [S^n, BG] \xrightarrow{ev_p} BG$ Action of this on

$[S^n, BG]_* = \pi_n BG = \pi_{n-1} G$

$G = U_1$, $BG = \mathbb{CP}^\infty \simeq K(\mathbb{Z}, 2)$. $[M, BU_1] \simeq [M, K(\mathbb{Z}, 2)]$.

In general, $[M, K(\pi, \ell)] = \prod_{k=0}^{\ell} K(H^{\ell-k}(M, \pi), k)$. (Thm of Thom)

$ev: M \times [M \times K(\pi, n)] \to K(\pi, n)$.

$\pi_0([M, K(\pi, \ell)]) = H^\ell(M, \pi)$.

$\pi_1([M, K(\pi, \ell)]) = \pi_0([\Sigma M, K(\pi, \ell)]) = H^\ell(\Sigma M, \pi) = H^{\ell-1}(M, \pi)$.

$H^\ell(K(\pi, \ell); \pi) = \mathrm{Hom}(\pi, \pi) \ni \mathbb{1}$. $ev^*(\mathbb{1}) \in H^\ell(M \times [M, K(\pi, \ell)], \pi)$

$a \in H_k(M, \pi)$. $\Rightarrow ev^*(\mathbb{1})/a \in H^{\ell-k}(M, \pi)$, i.e. an $[M, K(\pi, \ell)] \to K(\pi, \ell - k)$.

$[M, K(\mathbb{Z}, 2)] \simeq \prod_{k=0}^{2} K(H^{2-k}(M; \mathbb{Z}), k) = K(H^2(M; \mathbb{Z}), 0) \times K(H^1(M; \mathbb{Z}), 1) \times K(H^0(M; \mathbb{Z}), 2)$.

Suppose M is connected. $\Rightarrow = H^2(M; \mathbb{Z}) \times (S^1)^{b^1(M)} \times \mathbb{CP}^\infty$.

iso classes of cx line bundles L

Suppose M is a cpt mfld.

$\mathcal{G} = [M, S^1]$, $\mathcal{G}^0 \subset \mathcal{G} \xrightarrow{ev_{x_0}} S^1$.

$\mathcal{A}_L$ = connections in L, affine space for $i\Omega^1(M; \mathbb{R})$.

$\mathcal{G}^0$ acts freely on connections $\Rightarrow \mathcal{A}/\mathcal{G}^0$ is a classifying space for $\mathcal{G}^0$.

~~If A is a connection, $\exists u \in \mathcal{G}$ so that F_{u^*A} is a harmonic form.~~

~~$F_{u^*A} = F_A + d(u^{-1}du)$~~ $A = A_0 + a$, $\exists\, u$ s.t. $u^*A \in \mathcal{S}_{A_0}$

$u^*A = A_0 + a \mid d^*a = 0$.

$= A_0 + a + u^{-1}du$. $u = e^{if}$

$= A_0 + a + i\,df$

$a = i(d(-f) + \text{harmonic} + d^* \text{2-form})$

For $U(1)$, $\mathcal{A}/\mathcal{G} = S_{A_0}/\tilde{\mathcal{G}}$, $\tilde{\mathcal{G}} = \{u \mid u^* S_{A_0} = S_{A_0}\}$

$u^*(A_0 + a) = A_0 + u^{-1}du + a \in S_{A_0} \Rightarrow d^*(u^{-1}du) = 0 \Rightarrow u$ is harmonic