Moduli spaces, 9/24/08

$A \in \mathcal{A}^p_k$, $(k+1)p > n$. $S_{A,\varepsilon} = \{A + a \mid d_A^* a = 0, \|a\|_{L^p_k} < \varepsilon\}$.

$m : S_{A,\varepsilon} \times \mathcal{G}^p_{k+1} \to \mathcal{A}^p_k$ $\Longrightarrow$ $\bar{m} : (S_{A,\varepsilon} \times \mathcal{G}^p_{k+1}) / \mathrm{Stab}(A) \to \mathcal{A}^p_k$

$(A+a, u) \mapsto u \cdot (A+a) = A + u^{-1} d_A u + u^{-1} a u$.

Fix ε sufficiently small. $\bar{m}$ is a diffeo onto a nbhd of $\mathcal{G} \cdot A$.

Remains to see that $\bar{m}$ is injective.

$A+a, A+b \in S_{A,\varepsilon}$. $u \cdot (A+a) = A + b \Rightarrow u \in \mathrm{Stab}(A)$ if $\varepsilon << 1$.

$u^{-1} a u + u^{-1} d_A u = b \Leftrightarrow d_A u = ub - au$. $d_A^* d_A u = -\cancel{*} d_A * (ub - au) = -* d_A (u * b - * a u)$

$= -*(d_A u * b + u \underbrace{d_A * b}_{0} - \underbrace{(d_A * a)}_{0} u \pm a d_A u) = -*(d_A u * b) \pm *(* a\, d_A u)$

$u \in L^p_{k+1}(X; \mathrm{End}(E))$.

Take inner product with u^0 = component of u L^2-orthogonal to $\ker(d_A)$.

$\Rightarrow \int |d_A u^0|^2 \leq \int |d_A u^0| \, |b| \, |u^0| + |d_A u^0| \, |u^0| \, |a| \leq \|d_A u^0\|_{L^2} (\|a\|_{L^n} + \|b\|_{L^n}) \|u^0\|_{L^{2n/n-2}}$

$L^2 \quad L^n \quad L^{2n/n-2}$

$d_A : L^2_1(X; \mathrm{End}(E)) \to L^2(X; \mathrm{End}(E))$ has closed range with f.d. kernel

$\Rightarrow \|u^0\|_{L^2_{1,A}} \leq C \|d_A u^0\|_{L^2}$ $\qquad C = \frac{\quad}{\sqrt{1+\lambda_1}}$ $\leftarrow$ first positive eigenvalue of Δ_A.

$\Uparrow$

$\lambda_1 \|u^0\|^2_{L^2} \leq \|d_A u^0\|^2_{L^2}$.

$L^2_{1,A} \hookrightarrow L^{2n/n-2}$, norm of embedding is indep of A.

<u>Kato Inequality:</u> If A is a metric connection in $E \to X$ and s is a section of E, then $|d|s|| \leq |\nabla_A s|$ a.e.

$\Rightarrow$ for example, $\|s\|_{L^2_{1,A}} \geq \| |s| \|_{L^2_1} \geq C_{sob} \|s\|_{L^{2n/n-2}}$

$\uparrow$ Sobolev embedding constant

⊛ $\Rightarrow \|d_A u^0\|^2_{L^2} \le C_{sob} \|d_A u^0\|_{L^2} \|u^0\|_{L^2_{1,A}} (\|a\|+\|b\|)_{L^n}$

$$\le C_{sob}(1+\lambda_1^{-1})^{1/2}\|d_A u^0\|^2_{L^2}(\|a\|+\|b\|)_{L^n}.$$

Observation: Thus m is injective on a small L^n-ball in the slice, not just the L^p_k-ball.

$$d_A^* d_A u = \underbrace{\pm d_A u * a \pm b * d_A u}_{L^{p}\ 2r}.$$

L^p_{-1}

Should try to control: $\|d_A^* d_A u\|_{L^p_{-1}} \; (= (L^q_1)^*)$

$L^{p} \times L^r \hookrightarrow L^p_{-1}$

$\Updownarrow$

$L^{p} \times L^r \times L^{p'}_1 \hookrightarrow L^1$

$\frac{1}{p}+\frac{1}{r}+-\frac{1}{n}\left(1-\frac{n}{p}\right)=1$

$r=n$

Observation: L^n-norm on 1-forms is scale-invariant.

$\delta_\lambda:\mathbb{R}^n\to\mathbb{R}^n$
$x\mapsto\lambda x$

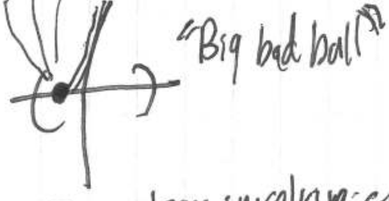
$a\in\Omega^1_c(\mathbb{R}^n)$. $\int|a|^n dvol = \int|\delta_\lambda^* a|^n dvol$

$$=\int_{\mathbb{R}^n}\lambda^n\left(\left(\Sigma|a_i(\lambda x)|^2\right)^{n/2} dx^1\wedge\cdots\wedge dx^n.\right.$$

$\tilde{S}_{A,\epsilon}=\{A+a \mid d_A^* a=0,\ a\in L^p_k,\ \|a\|_{L^n}<\epsilon\}$.

$m:\tilde{S}_{A,\epsilon}\times \mathcal{G}^p_{k+1}\to \mathcal{A}^p_k$.

Prop: $m:(\tilde{S}_{A,\epsilon}\times \mathcal{G}^p_{k-1})/_{Stab(A)}\longrightarrow\mathcal{A}^p_k$ is a diffeo onto its image, and its image contains an L^n-ball about A in $\mathcal{A}^p_k$.



$A\in\mathcal{A}^p_k$, $F_A=dA+A\wedge A$ locally, in L^p_{k-1}, $(k+1)p>n$.

Equations involving curvature: $\mathcal{E}(A)=\int_M|F_A|^2 dvol$. Sometimes might consider $\mathcal{E}^p(A)=\int|F_A|^p dvol$.

$F_{u\cdot A}=u^{-1}F_A u \Rightarrow \mathcal{E}^p(u\cdot A)=\mathcal{E}^p(A)$.

When $n=2$ or 4, this functional has particularly nice properties.

When $n=4$, YM is scale invariant, because L^2-norm on 2-forms is scale-invariant. The $L^{n/k}$ norm on k-forms is scale invariant in dimension n.

Explicitly, $\mathcal{E}(A) = \int_A -\mathrm{tr}(F_A \wedge *_g F_A)$. Conformally invariant in dim 4, $g \mapsto e^{2\sigma} g$, $\mathcal{E}$ doesn't change.

$$\frac{d}{dt}\mathcal{E}(A+ta)\Big|_{t=0} = 0 \Rightarrow 0 = \frac{d}{dt}\int_M -\mathrm{tr}(F_{A+ta} \wedge * F_{A+ta})\Big|_{t=0}$$

$$= \frac{d}{dt}\int -\mathrm{tr}(F_A + d_A a + t^2 a\wedge a \wedge * F_A + t d_A a + t^2 a \wedge a)\Big|_{t=0}$$

$$= -2\int \mathrm{tr}(F_A \wedge * d_A a) = 0 \quad \forall a$$

$$= \langle F_A, d_A a\rangle_{L^2} \Rightarrow d_A^* F_A = 0. \quad \text{Yang-Mills Equation.}$$

Observe that Bianchi identity is $d_A F_A = 0$.

$\Rightarrow$ Curvature of a Y-M connection is like a "harmonic" 2-form.

local YME $d^*_{a^\gamma}(da^\gamma + a^\gamma \wedge a^\gamma) = 0$. $\quad d^* d a^\gamma + \mathrm{quad}(a^\gamma, \nabla a^\gamma) + \mathrm{cubic}(a^\gamma, a^\gamma, a^\gamma) = 0$

Bad because linearized eqn is not elliptic. d^*d on Ω^1 is not elliptic for $n \geq 2$.

But: Slice condition $d^* a^\gamma + A^\gamma a^\gamma = 0$ locally.

~~$dd^* a^\gamma + \mathrm{quad}(a^\gamma, \nabla a^\gamma) = 0$~~

When $n=2$, $*F_A \in \Omega^0(M, \mathrm{End}(E))$. $\Rightarrow *F_A$ is parallel.

$E \to \Sigma^2$ vector bundle, ξ a parallel section of $\mathfrak{u}(E) \to \Sigma^2$ Skew-Hermitian endomorphism

$\mathrm{Spec}(\xi)$ is $\{\sqrt{-1}\lambda_i\}$, $\lambda_i \in \mathbb{R}$.

Eigenspaces $E = \oplus E_i$, $\xi|_{E_i} = \sqrt{-1}\lambda_i$. If e_λ is a local eigenvector section of E,

$\xi e_\lambda = i\lambda e_\lambda$. $\nabla^A_X(\xi e_\lambda) = \xi \nabla^A_X e_\lambda = i\lambda \nabla^A_X e_\lambda \Rightarrow \nabla^A$ preserves E_i.

Powers of ξ are parallel, $\mathrm{Tr}(\xi^k)$ are constant $\Rightarrow$ eigenspaces are constant.

Locally ξ becomes a family of skew-hermitian matrices w/ constant eigenvalues.

Solutions correspond to projectively flat connections.