$\mathcal{A}^p_k \ni A \quad S_{A,\varepsilon} = \{A + a \mid d_A^* a = 0,\ \|a\|_{L^p_k} < \varepsilon\}$

$m : (S_{A,\varepsilon} \times \mathcal{G})/\mathrm{Stab}(A) \to \mathcal{A}^p_k$ is a diffeo onto a nbhd of $\mathcal{O}_A$.

$A \in \mathcal{A}^p_k$. Define $\|\xi\|^p_{j,A} = \sum_{i=0}^{j} \int_X |\nabla_A^j \xi|^p \, d\mathrm{vol}$.

If $(k+1)p > n \Rightarrow \|\xi\|_{L^p_{j,A}} \sim \|\xi\|_{L^p_{j,A_0}}$ where A_0 is a C^∞-connection

$A = A_0 + a$. Want: $\int \left(|(\nabla_{A_0} + a)^k \xi|^p + \dots + |\xi|^p \right) d\mathrm{vol} \sim \int \left(|\nabla_{A_0} \xi|^p + \dots + |\xi|^p \right) d\mathrm{vol}$

Want: $\|\nabla_{A_0+a} \xi\|_{L^p_{j,A_0}} \le \|\xi\|_{L^p_{j+1,A_0}} \left(1 + \|a\|_{L^p_{k,A_0}}\right)$

$$\wedge\!|$$

$\|\nabla_{A_0} \xi\|_{L^p_{j,A_0}} + \|a \xi\|_{L^p_{j,A_0}} \le \|\xi\|_{L^p_{j+1,A}} + \|a\|_{L^p_{k,A_0}} \|\xi\|_{L^p_{j+1,A_0}}$

$(j - \frac{n}{p}) \le j + 1 - \frac{n}{p} + k - \frac{n}{p} \Rightarrow (k+1)p > n$

Same as condition for gauge group to act continuously.

$\mathcal{G}^p_{k+1} \subset L^p_{k+1}(X; \mathrm{End}(E))$. $\quad L^p_{k+1,A}$ norms are all equivalent for $A \in \mathcal{A}^p_k$.

$m : S_{A,\varepsilon} \times \mathcal{G}^p_{k+1} \to \mathcal{A}^p_k$

$(A + a, u) \mapsto u \cdot (A + a) = A + u^{-1} d_A u + u^{-1} a u$ Smooth map of Banach mflds

Step 1: Use IFT to prove local diff

The differential of m

$$\mathcal{D}_{(A,1)} m : \ker\left(d_A^* : L^p_{k,A} \to L^p_{k-1,A}\right) \oplus L^p_{k+1}(X; \mathrm{ad}\, P) \to L^p_{k,A}(X; T^*X \otimes \mathrm{ad}\, P)$$

$$(\alpha, \xi) \mapsto d_A \xi + \alpha$$

Need that ⓐ $\ker(\mathcal{D}_{(A,1)} m) = \{(0,\xi) \mid d_A\xi = 0\}$, and ⓑ $\mathcal{D}_{(A,1)} m$ is surjective.

ⓐ $d_A\xi + a = 0 \Rightarrow d_A\xi = 0,\ a = 0$, but $d_A\xi$ and a are L^2-orthogonal

$\Rightarrow \langle d_A\xi, d_A\xi + a\rangle = \|d_A\xi\|^2_{L^2} \Rightarrow d_A\xi = 0$, etc

ⓑ Note ~~that~~ $\mathrm{Im}(d_A : L^p_{k+1,A} \to L^p_{k,A})$ is closed.

$\|d_A\xi\|_{L^p_{k,A}} \leq \|\xi\|_{L^p_{k+1,A}} \Rightarrow$ closed range and f.d. kernel by the functional analysis result

~~Need $\|d_A$~~

$B' \subset B$ cpt $\quad T : B' \to B$ bounded operator.

If $\|Tx\|_B + \|x\|_B \geq \|x\|_{B'} \Rightarrow$ f.d. ker and closed range

Need $\|d_A\xi\|_{L^p_{k,A}} + \|\xi\|_{L^p_{k,A}} \geq C(\|\xi\|_{L^p_{k+1,A}})$

<u>Thm:</u> $L^2(X; T^*X \otimes \mathrm{ad}\, P) = \mathrm{Im}(d_A : L^2_1 \to L^2) \oplus \ker(d_A^* : L^2 \to L^2_{-1})$, $L^2_{-1} = (L^2_1)^*$

<u>Thm:</u> $a \in L^p_{k,A}$, $a = d_A\tilde{\xi} + \tilde{\alpha}$, $d_A\tilde{\xi} \in L^2$, $\tilde{\alpha} \in L^2$

To show: $\tilde{\xi}$ can be chosen to be in $L^p_{k+1,A}$, $\tilde{\alpha} \in L^p_k$.

$\underbrace{d_A^* a}_{L^p_{k-1,A}} = d_A^* d_A \tilde{\xi} \qquad$ Need to check elliptic regularity holds for $d_A^* d_A$.

Assume elliptic regularity for $d_{A_0}^* d_{A_0}$ is an index zero Fredholm operator for every p, k.

$$d_A^* d_A : L^p_{j,A} \to L^p_{j-2,A} \quad j \le k+1$$ is Fredholm of index zero.

$$d_A^* d_A - d_{A_0}^* d_{A_0} = \left((d_{A_0}+a)^* (d_{A_0}+a) - d_{A_0}^* d_{A_0} \right) \xi$$

$$= \underset{①}{a^* \cdot d_{A_0}\xi} + (\nabla_{A_0} a)\xi + \underset{①}{a \cdot d_{A_0}\xi} + a^* a\,\xi.$$

Want to show that this is compact.

If a is smooth, then we're in L^p_{k-1}, so we're all set.

$$\cancel{L^p_{k,A} \times L^p_{k,A} \to L^p_{k-2,A}. \quad (k - n/p) + (k - n/p) \ge k - 2 - n/p}$$

① $L^p_{k,1} \times L^p_{j-1,A} \to L^p_{j-2}, \quad j \le k+1$

$$(k - n/p) + (j - 1 - n/p) > (j-2) - n/p$$

$$k + 1 - n/p > 0 \quad ✓$$

② $L^p_{k-1,A} \times L^q_{j,A} \to L^r_{j-2,A}$ ok

③ $L^p_{k,A} \times L^p_{k,A} \times L^p_j \to L^p_{j-2}$

$$(k - n/p) + (k - n/p) + (j - n/p) > (j-2) - n/p$$

$$2((k+1) - n/p) > 0 \quad ✓$$

$$\Longrightarrow \| (d_A^* d_A - d_{A_0}^* d_{A_0})\xi \|_{L^p_{j-2,A}} \le f(\|a\|_{L^p_{k,A}}) \|\xi\|_{L^p_{j,A}}$$

Compact for smooth a. Approximate a by something smooth.

If $A_i \in C^\infty \xrightarrow{L^p_k} A$, then this estimate implies that $\|\delta(A, A_i)\|_{Op(L^p_j, L^p_{j-2})} \to 0$

but $d_{A_i}^* d_{A_i} - d_{A_0}^* d_{A_0}$ are cpt $\Longrightarrow$ (closedness of compact operators in operator topology) $d_A^* d_A - d_{A_0}^* d_{A_0}$ cpt.

$\Rightarrow d_A^* d_A : L^p_{j,A} \to L^p_{j-2,A}$ is Fredholm index zero for $j \le k+1$.

$\Rightarrow \|\xi\|_{L^p_{j,A}} \le c\left(\|d_A^* d_A \xi\|_{L^p_{j-2,A}} + \|\xi\|_{L^p}\right).$

$d_A^* a = d_A^* d_A \tilde{\xi} \in L^p_{k-1} \Rightarrow \tilde{\xi} \in L^p_{k+1} \Rightarrow \tilde{\alpha} \in L^p_k.$

L^2: Weitzenbock
Fourier transforms

L^p elliptic thy: Taylor PDE
Stein

Paley-Littlewood: understand Fourier transforms

Stein: Interpolation result

$\boxed{p \ne 1, \infty \text{ for elliptic regularity}}$

$\downarrow$

Replacements: BMO

$\|f\|_{BMO} = \lim_{\text{radius}(B) \to 0} \inf \frac{\int_B |f - \bar{f}_B|}{\mathrm{vol}(B)}$ $\quad \bar{f}_B = \frac{\int_B f}{\mathrm{vol}\,B}$ $\quad B \subset \mathbb{R}^n$.

$\partial_i \partial_j G : L^\infty \not\to L^\infty$
$\quad BMO \to BMO$

$L^n_1 \hookrightarrow BMO$
$L^n_1 \not\hookrightarrow L^\infty$

BMO maps between spheres have degree

Thm: $\bar{m} : (S_{A,\varepsilon} \times G)/\mathrm{Stab}(A) \to \mathcal{A}^p_k$

$T_{[A,1]}(S_{A,\varepsilon} \times G)/\mathrm{Stab}(A) = \left(\ker(d_A^* : L^p_k \to L^p_{k+1}) \oplus L^p_{k+1}(X; \mathrm{ad}\,P)\right) / \ker d_A :$

$D\bar{m}_{[A,1]} : T_{[A,1]}(S_{A,\varepsilon} \times G^p_{k+1})/\mathrm{Stab}(A) \to T_A \mathcal{A}^p_k$ is an isomorphism $\quad L^p_{k+1} \to L^p_k$).

$\Rightarrow \bar{m}$ is a local diffeo for some $\varepsilon < 0$.

Via equivariance for ε sufficiently small, $\bar{m}$ is a local diffeo. Remains to show injectivity.

$A+a, A+b \in S_{A,\varepsilon}$ for ε small enough

WTS $u\cdot(A+a) = A+b \Rightarrow u \in \mathrm{Stab}(A)$

$u^{-1} d_A u + u^{-1} a u = b.$ $\quad d_A u = ub - au$

$d_A^* d_A u = -*d_A*(ub - au) = -*d_A(u(*b) - (*a)u)$

$= -(*(d_A u \wedge *b) \pm *a\, d_A u)$ inner product of both sides with u...