$\mathcal{B}^P_k = \mathcal{A}^P_k / \mathcal{G}^P_{k+1}$ $\qquad S_{x_0} = \exp\left(u \subset (T_{x_0}\mathcal{O}_{x_0})^\perp\right)$

$(S_{x_0} \times \mathcal{G})/\mathrm{Stab}(x_0) \to X$ is a diffeo onto nbhd of $\mathcal{O}_{x_0}$.

Understand $\mathrm{Stab}(A)$, $A \in \mathcal{B}^P_k$

$G = U(n)$ or a subgroup, P = unitary frames of a $U(n)$ bundle $E \to X$

u is a section of $\mathrm{Ad}P \subset \mathrm{End}(E)$, $\quad u^* u = 1$.

$\mathrm{Ad}P \to X$, $\quad \mathrm{End}(E) \to X$

$u^{-1} d_A u = d_A$

$= d_A + u^{-1} d_A u$

$u^{-1} d_A u = 0 \iff d_A u = 0$.

$\mathrm{Stab}(A) \subset \mathcal{G}$. Given $u \in \mathrm{Stab}(A)$, $p_0 \in P$, $\tilde{u}(p_0) \in U(n)$ $\quad ev: \mathrm{Stab}A \subset U(n)$

Groups associated to a connection: ① Stabilizer ② Holonomy

Given γ a loop based at x_0, we can lift γ to $\tilde{\gamma}$ a path starting at p_0 and ending at $p_0 g$. $g = hol_{\gamma, p_0}(A)$.

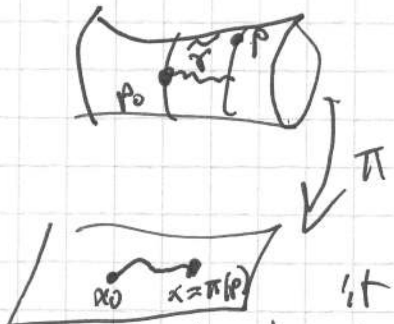
$hol_{p_0}(A) = \bigcup_{\gamma \in \Omega_{x_0}(X)} hol_{\gamma, p_0}(A) \subset G = U(n)$ <u>is a subgroup</u>

Prop: $\mathrm{Stab}_{p_0}(A)$ = Centralizer of $hol_{p_0}(A) = \{ g \in G \mid g^{-1} h g = h \ \forall\, h \in hol_{p_0}(A) \}$.

1) $\mathrm{Stab}_{p_0}(A) \subset \mathrm{Cent}(\mathrm{hol}_{p_0}(A))$ follows from the fact that each $g \in \mathrm{Stab}_{p_0}(A)$ extends by definition to $\tilde{g} \in \mathcal{G}$, $\tilde{g} \cdot A = A$

2) $\mathrm{Cent}(\mathrm{hol}_{p_0}(A)) \subset \mathrm{Stab}_{p_0}(A)$.

$g \in \mathrm{Cent}(\mathrm{hol}_{p_0}(A))$ Try to extend g to a gauge transformation $\tilde{g}$ by the rule $\tilde{g}(p) = kgk^{-1}$ if γ is a path from x_0 to x with A-parallel lift $\tilde{\gamma}$ having $\tilde{\gamma}(0) = p_0$, $\tilde{\gamma}(1)k = p$.



$\Rightarrow k'k^{-1} \in \mathrm{hol}_{x,p}(A) \Rightarrow kgk^{-1} = h'g(h')^{-1}$.

Why is it smooth? To check suffices to do it locally. Use parallel transport with A along radial geodesics in a local chart to trivialize the bundle. In the local trivialization, $\tilde{g}$ is constant.

Only subgroups that appear as centralizers can be stabilizers!

E.g.

G	centralizers
$U(1)$	$U(1)$
$SU(2)$	$\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$, ± 1, $SU(2)$
SO_3	SO_2, $\mathbb{Z}_2^2$, $\mathbb{Z}_2$, $O(2)$
U_n	decompose $\mathbb{C}^n = \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_k}$; $\begin{pmatrix} U(n_1) & & & & \\ & U(1) & & & \\ & & \ddots & & \\ & & & U(1) & \\ & & & & U(n_k) \end{pmatrix}$

$\mathbb{Z}_2 = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \right\} \underset{\leftarrow}{\overset{\mathrm{Cent}}{\rightarrow}} \begin{pmatrix} A & 0 \\ 0\ 0 & \det A \end{pmatrix}$, $A \in O(2)$

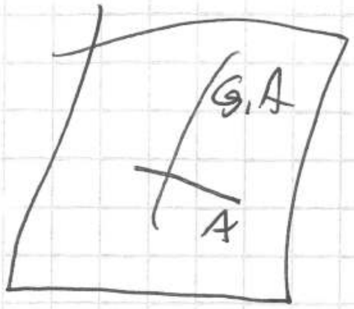
$\mathbb{Z}_2^2 \sim \begin{pmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{pmatrix}$ $\varepsilon_1 \varepsilon_2 \varepsilon_3 = 1$ centralizes itself

Centralizers are a directed set where centralizing is an involution

Centralizer of centralizer

Connection A reduces structure group to a $\mathrm{hol}_{p_1}(A)$-bundle.

Given $G' \subset G$ a possible stabilizer we can construct a G'-bundle with connection, so that the holonomy of the connection is G'.



Put a Riemannian structure on $\mathcal{A}^p_k$ by $T_A \mathcal{A}^p_k := L^p_k(X, T^*X \otimes \operatorname{ad} P)$.

$$\langle \alpha_1, \alpha_2 \rangle = \int_X (\alpha_1 \wedge * \alpha_2).$$

$(\, , \,)$ is a bi-invariant product on $\mathfrak{g}$.

$\alpha_1 = \sum \theta_{1i} \otimes \xi_1^i$, $\theta_{1i} \in \Gamma(U, T^*X)$

$\alpha_2 = \sum \theta_{2i} \otimes \xi_2^i$. $\xi_1^i \in \Gamma(U, \operatorname{Ad} P)$

$$(\alpha_1 \wedge * \alpha_2) = \sum_{i,j} \theta_{1i} \wedge * \theta_{2j} \, (\xi_1^i, \xi_2^j).$$

Results in an n-form.

$G \cdot A = \{ d_A + u^{-1} d_A u \mid u \in G \}$

$T_A(G \cdot A) = \{ d_A \xi \mid \xi \in \Gamma(X, \operatorname{ad} P) \}$.

$$= \frac{d}{dt} \left(d_A + \exp(t\xi)^{-1} d_A \exp(t\xi) \Big|_{t=0} \right)$$

$\exp : \operatorname{ad} P \to \operatorname{Ad} P$

fiberwise exponential map

$\xi \in \Gamma(X, \operatorname{ad} P)$

$\exp(\xi) \in \Gamma(X; \operatorname{Ad} P)$.

$$T_A \mathcal{O}_A^{\perp} = \left\{ \alpha \in \Gamma(X; \Lambda^1 \otimes \operatorname{ad} P) \;\Big|\; \int_X (d_A \xi \wedge * \alpha) = 0 \;\; \forall \xi \in \Gamma(X; \operatorname{ad} P) \right\}$$

$$\int_X (d_A \xi \wedge * \alpha)$$

$$0 = \int_X d(\xi \wedge * \alpha) = \int_X d_A \xi \wedge * \alpha = \int_X \xi \wedge d_A * \alpha$$

$$\Rightarrow = -\int_X \xi \wedge d_A * \alpha = \pm \int_X (\xi, * (* d_A * \alpha)).$$

$d_A^* = -* d_A *$, and $T_A \mathcal{O}_A^{\perp} = \{ \alpha \mid d_A^* \alpha = 0 \}$.

Try to show that: $S_{A,\varepsilon} = \{ A + \alpha \mid d_A^* \alpha = 0, \ \|\alpha\|_{L^p_k} < \varepsilon \}$

is a slice.

Prop: For ε sufficiently small, $(\mathcal{G} \times S_{A,\varepsilon})/\mathrm{Stab}(A)$ is diffeomorphic to a nbhd of $\mathcal{O}_A$ in $\mathcal{A}^p_k$.

$d_A^*(\nabla_A^* \nabla_A + 1)\alpha = 0$ if we used the L^2_1 inner product.
Doesn't work well because it requires too much regularity on A.

$$m: S_{A,\varepsilon} \times \mathcal{G}^p_{k+1} \to \mathcal{A}^p_k$$
$$(A+\alpha, g) \mapsto g \cdot (A+\alpha)$$

Equivariance: $m(A+\alpha, hg) = h\, m(A+\alpha, g)$.

Step 1: Show that $d_{(A,1)} : T_A S_{A,\varepsilon} \times T_I \mathcal{G}^p_{k+1} \to T_A \mathcal{A}^p_k$ is surjective with kernel $\mathrm{Lie}(\mathrm{Stab}(A))$ in $T_A S_{A,\varepsilon} \times T_I \mathcal{G}^p_{k+1}$

If it's a local diffeo at identity, then by equivariance it's a local diffeo along the orbit.

Step 2: Observe that IFT $\Rightarrow$ induced map $\bar{m} : (S_{A,\varepsilon} \times \mathcal{G}^p_{k+1})/\mathrm{Stab}(A) \to \mathcal{A}^p_{k+1}$ is a local diffeo at $[A, 1]$

Step 3: Observe that 2 + equivariance of $m \Rightarrow \bar{m}$ is a local diffeo in a nbhd of orbit.

Step 4: Prove $\bar{m}$ is injective.