9/10/08 Moduli spaces

$$\mathcal{A}^p_k \hookrightarrow \mathcal{A}^q_\ell \overset{?}{\Rightarrow} \mathcal{B}^p_k \hookrightarrow \mathcal{B}^q_\ell$$

Suppose $u \in \mathcal{G}^\infty_0 = \{u \in L^\infty(X; \operatorname{End} E) \mid u^* u = 1 \text{ a.e.}\}$

$\exists A, \tilde{A} \in \mathcal{A}^p_k$ s.t. $\circledast\ u \cdot A = \tilde{A} \Rightarrow u \in \mathcal{G}^p_{k+1}$.

In a local trivialization, $\circledast \Rightarrow u^{-1} du + u^{-1} a^\tau u = \tilde{a}^\tau$

$\Rightarrow du = u\tilde{a}^\tau - a^\tau u$. If $\tilde{a}^\tau, a^\tau \in L^p \overset{\circledast}{\Longrightarrow} u \in L^p_1$

$\tilde{a}, \tilde{a}^\tau \in L^p_1$. $\quad \nabla du = \underbrace{u \nabla \tilde{a}^\tau}_{L^p} - \underbrace{\nabla a^\tau u}_{L^p} + \underbrace{\nabla u\, \tilde{a}^\tau}_{\substack{L^p L^p_1 \\ L^n}} - a^\tau \nabla u$

If $p \le n$, done. If $p \ge n$, then $\nabla du \in L^n$

$\Rightarrow u \in L^n_2$. $\nabla u \in L^n_1 \hookrightarrow L^p\ \forall p$

$\Rightarrow \nabla u\, \tilde{a}^\tau \in L^1$ (in fact better)

$p(k+1) > n$ is Hausdorff.

Consider $\mathcal{A}^n_0$ $\mathcal{G}^n_1$ Action not cts

Prop: The $\mathcal{G}^n_1$ orbits are weakly (hence strongly) closed in $\mathcal{A}^n_0$.

Suppose $A \in \mathcal{A}^n_0$, $u_i \in \mathcal{G}^n_1$, $u_i \cdot A \xrightarrow{L^n} \tilde{A}$. Want to show $\exists u \in \mathcal{G}^n_1$ s.t. $u \cdot A = \tilde{A}$.

Proof: $u_i^{-1} du_i + u_i^{-1} a^\tau u_i = \tilde{a}^\tau_i \xrightarrow{L^n} a^\tau$.

$du_i = u_i \tilde{a}^\tau_i - a^\tau u_i$ is uniformly bounded in L^n.

$\Rightarrow u_i \xrightarrow{L^n_1} u \Rightarrow u_i \xrightarrow{L^p} u$.

$u \in L^n_1(X; \mathrm{End}(E))$. $u_i \xrightarrow{L^p} u \Rightarrow u_i \to u$ a.e. $\Rightarrow u^* u = 1$ a.e. $\Rightarrow u \in \mathcal{G}^n_1$.

$$\begin{array}{ccc} du_i & = u_i \tilde{a}^\tau_i & - a^\tau u_i \\ \downarrow & & \downarrow \\ du & & a^\tau u \end{array}$$

Pass to a subsequence where $u_i \tilde{a}^\tau_i$ converges weakly to $\tilde{\tilde{a}}$.

WTS $u \tilde{a}^\tau = \tilde{\tilde{a}}$. Enough to check in $L^{n-\varepsilon}$.

Lemma: If $a_i \in L^p_1$, $b_i \in L^q_1$, $a_i \xrightarrow{L^p} a$, $b_i \xrightarrow{L^q} b \Rightarrow a_i b_i \xrightarrow{L^r} ab$

$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Apply this Lemma to $u_i \in L^{\text{very big}}_1$, $\tilde{a}^\tau_i \in L^n_1$, $u_i \tilde{a}^\tau_i \xrightarrow{L^r} u \tilde{a}^\tau$

Uhlenbeck: (L^p-bounds on curvature, removable singularities for YM.)

Prop: $\mathcal{B}^n_0$ is a metric space. $d([A],[\tilde{A}]) = \inf_{u, \tilde{u} \in \mathcal{G}^n_1} \|uA - \tilde{u}\tilde{A}\|_{L^n(X)}$.

$u \cdot A - u \cdot \tilde{A} = u \cdot (A - u^{-1} \tilde{u} \tilde{A})$. u is an isometry of L^n, so

$= \inf_{u \in \mathcal{G}} \|A - u\tilde{A}\|_{L^n(X)}$. d satisfies triangle inequality: (using triangle inequality for L^n.

Prop: $d([A],[\tilde{A}]) = 0 \Rightarrow [A] = [\tilde{A}]$

Proof: $\Rightarrow \exists u_i$ st. $\lim_{i\to\infty} \|u_i A - \tilde{A}\|_{L^n} \to 0 \Rightarrow u_i A$ is in some L^n ball about $\tilde{A}$

$\Rightarrow u_i A$ converges weakly to $\tilde{\tilde{A}}$ (after passing to subseq).
$\Rightarrow \tilde{\tilde{A}} = uA$ for some $u \in \mathcal{G}^n_1$ (by weak closure of orbits).

$\lim \|u_i A - \tilde{A}\| = 0 \Rightarrow \tilde{\tilde{A}} = \tilde{A} = uA$.

Why do you care that orbits are weakly closed?
<u>Ans:</u> Exactly what you need to prove that this distance is a metric.

Hope to carry over the pictures from finite dimensions

X smooth, G cpt Lie group acting smoothly on X. Local structure of X/G?

$\mathbb{C}^n/(S^1)^n = (\mathbb{R}^+)^n$ $\qquad$ $x_0 \in X$, investigate nbhd of orbit $\mathcal{O}_{x_0} = Gx_0$.

$\mathrm{Stab}_{x_0} = \{g \in G \mid gx_0 = x_0\}$. $\quad \mathcal{O}_{x_0} = G/\mathrm{Stab}_{x_0}$ ← closed Lie subgroup

Take a G-invariant Riemannian metric on X.

$T_{x_0}X = T\mathcal{O}_{x_0} \oplus (T\mathcal{O}_{x_0})^\perp$. $\quad \exp_{x_0} : T_{x_0}X \to X$

We can embed a Stab_{x_0}-invariant nbhd of $0 \in (T\mathcal{O}_x)^\perp$ into X giving a submanifold S_{x_0} called a slice.

<u>Prop:</u> $S_{x_0} \times G \to X$ $\qquad$ Stab_{x_0} acts on $S_{x_0} \times G$
$(x, g) \to xg$

$h \in \mathrm{Stab}_{x_0}$, $\quad h \cdot (x, g) = (xh, h^{-1}g)$.

$\Upsilon : S_{x_0} \times G / \mathrm{Stab}_{x_0} \to X$. $\quad \Upsilon$ is diffeomorphism onto a nbhd of $\mathcal{O}_{x_0}$.

$\Rightarrow$ Nbhd of $[x_0] \in X/G$ is homeomorphic to $S_{x_0}/\mathrm{Stab}_{x_0}$.

Given a connection A, what's the stabilizer in $\mathcal{G}$?

$u^{-1} d_A u = 0$ $\boxed{d_A u = 0}$

Examples:

$$\begin{array}{c} S^1 \to P \\ \downarrow \\ X \end{array}$$

$\mathrm{End}(E) = \mathbb{C}$

$u : X \to S^1$,

$d_A u = du = 0$

u is constant.

Example:

$$\begin{array}{c} \mathbb{C}^n \to E \text{ hermitian} \\ \downarrow \\ X \end{array}$$

$P = Fr_{U(n)} E$.

If A is a connection on E, $\mathrm{End}(E) = \mathrm{End}_0(E) \oplus \underset{\substack{\uparrow \\ tr}}{\mathbb{C} \cdot \mathbb{1}}$

$$\begin{array}{c} G \to P \\ \downarrow \\ X \end{array}$$

$\mathrm{Aut}(P) = \{ \hat{u} : P \to G \mid \hat{u}(pg) = g^{-1} \hat{u}(p) g \}$ $\nabla_A \otimes 1 + 1 \otimes \nabla_A$

If $z \in Z(G)$, then $\hat{u}_z : P \to G$ given by $\hat{u}_z(p) = z$. This is always parallel.

Covariantly constant gauge is determined (via parallel transport) from one point. $\mathrm{Stab}_A \subset \mathcal{G}$ is a closed Lie subgroup of $\mathcal{G}$.