Sobolev spaces: $L^p_k(X) \hookrightarrow L^q_\ell(X)$ if $k \geq \ell$ and $(k - \frac{n}{p}) \geq (\ell - \frac{n}{q})$ ← conformal weight

Furthermore, if both are strict, then embedding is compact.

Multiplication: $L^p_k \times L^q_\ell \to L^r_m$.

a) If $(k - \frac{n}{p}) + (\ell - \frac{n}{q}) \geq m - \frac{n}{r}$, $k, \ell \geq m$

b) If $(k - \frac{n}{p}) > 0$ and $\ell - \frac{n}{q}$, $k - \frac{n}{p} \geq m - \frac{n}{r}$, $k, \ell \geq m$. $\ell - \frac{n}{q}$

If $f \in L^p_k$ and $g \in L^q_\ell$, $\nabla^m(f \cdot g) = \sum c_i \nabla^i f \nabla^{m-i} g$.

$\|\nabla^i f \nabla^{m-i} g\|_{L^r} \leq \|\nabla^i f\|_{L^{r'}} \|\nabla^{n-i} f\|_{L^{r''}}$, $\frac{1}{r} \geq \frac{1}{r'} + \frac{1}{r''} \Rightarrow m - \frac{n}{r} \leq (i - \frac{n}{r'}) + (m+i) - \frac{n}{r''}$ (?)

b) $\|\nabla^r g f\|_{L^m} \leq \|\nabla^r g\|_{L^m} \|f\|_{L^p_k} \leq \|g\|_{L^q_\ell} \|f\|_{L^p_k}$

$g \in C^0$

$\mathcal{A}^p_k = \{A \mid$ if $\tau: \pi^{-1}(U) \to U \times P$ is a C^∞ trivialization and $\varphi \in C^\infty_0(U)$, we have $a_\tau \in \Omega^1(U; \mathfrak{g})$, $\varphi a_\tau \in L^p_k(U; T^*U \otimes \mathfrak{g})\}$

$A \to F_A$ curvature in $\Omega^2(X; \operatorname{ad} P)$

$a^\tau \mapsto da^\tau + [a^\tau \wedge a^\tau]$. When is $F: \mathcal{A}^p_k \to L^q_\ell(X; \Lambda^2 \operatorname{ad} P)$ a cts map?

$a^\tau \mapsto da^\tau$ is cts into L^p_{k-1}.

$a^\tau \mapsto [a^\tau \wedge a^\tau]$ " " " " when $2(k - \frac{n}{p}) \geq (k-1) - \frac{n}{p} \Leftrightarrow k+1 \geq \frac{n}{p}$

(unless $k = \frac{n}{p}$)

$L^n_1 \hookrightarrow L^4$

$a^\tau \mapsto [a^\tau \wedge a^\tau]$ is cts. **Def.** X, Y Banach spaces, $U \subset X$ open, $f: U \to Y$.

We say f is differentiable at $x \in U$ if $\exists A: X \to Y$ a bounded linear operator and $C > 0$, $\|f(x-h) - f(x) - Ah\|_Y \leq C\|h\|^2_X$. ← Bad definition, but okay. (Do directional derivatives exist? Uniformly?)

we say f is cts diff on U if $x \mapsto A_x = d_x f$, $U \to \operatorname{Hom}(X, Y)$ is a cts map.

f is twice differentiable if cts diffible in a nbhd of x, and $x \mapsto d_x f$ is differentiable at x.

Multiplication is smooth if cts. (Higher derivatives vanish)

$\mu : (f,g) \mapsto fg$, $\quad d_{(f,g)}\mu(\delta f, \delta g) = f\delta g + \delta f g$.

$\| \mu(f+\delta f, g+\delta g) - \mu(f,g) - d\mu_{(f,g)}(\delta f, \delta g) \| = \|\delta f \delta g\|_{L^p_k} \le C(\|\delta f\|_{L^p_k}\|\delta g\|_{L^p_k})^2$.

What does $\mathcal{G}_{L^p_k}$ mean? Specialize to $G = O(n)$ or $U(n)$ or $Sp(n)$.

P = frame bundle of some vector bundle: E

u is a smooth gauge transform, $u : E \to E$ over X, u is a unitary iso on each fiber.

$u \in C^\infty(X; \mathrm{End}(E)),\ u^*u = 1)$.

<u>Def:</u> $\mathcal{G}^p_k = \{u \in L^p_k(X; \mathrm{End}\,E) \mid u^*u = 1 \text{ a.e.}\}$.

When is $\mathcal{G}^p_k$ a topological group? A group? A Banach Lie group?

$u, v \in \mathcal{G}^2_1 \Rightarrow u, v \in L^\infty$.

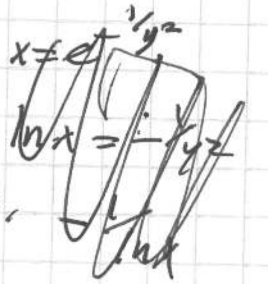
$\nabla(uv) = \nabla u\, v + u \nabla v \overset{?}{\in} L^2$

$L^2 L^\infty\ L^\infty L^2 \Rightarrow \nabla(uv) \in L^2 \Rightarrow \mathcal{G}^2_1 \times \mathcal{G}^2_1 \to \mathcal{G}^2_1$ is a group.

(but not a topological group with the L^2_1-topology.)

<u>Prop:</u> If $(k - \frac{n}{p}) > 0$ then $\mathcal{G}^p_k$ is a Banach Lie group.

$L^p_k(X; \mathrm{End}(E))$ is a Banach space which is an algebra. $\Phi : s \mapsto s^*s$.



$$L^p_k(X; \mathrm{End}\,E) \xrightarrow{\Phi} L^p_k(X; \mathrm{End}_{sA}(E)) \subset L^p_k(X; \mathrm{End}\,E).$$

Φ is a C^∞ map. $d_s\Phi(t) = s^*t + t^*s$. $\forall s \in \mathcal{G}^p_k$, $d_s\Phi$ is a surjection.

$d_s\Phi(\frac{1}{2}sw) = w$, $w \in \mathrm{End}_{sA}(E)$, $\ker(d_s\Phi) = \{sr \mid r = -r^*\}$ $\therefore$ closed complement!

The implicit function theorem holds, so $\Phi^{-1}(1) = \mathcal{G}^p_k$ is a Banach Lie group.

IFT needs surjective differential, and $\ker(d\Phi)$, must have a closed complement

When does $\mathcal{G}^p_k$ act on $\mathcal{A}^p_\ell$? $(u, A) \mapsto uA$ $\begin{cases} d_A + u^{-1}d_A u \\ u^{-1}du + u^{-1}a^? u. \end{cases}$

Need L^q_e and L^p_{k-1} to be a module for L^p_k.

For L^p_{k-1} to be a module for L^p_k, we need $(k-\frac{n}{p})>0$.

$\mathcal{G}^1_k \times \mathcal{A}^p_{k-1} \to \mathcal{A}^p_{k-1}$ is smooth for $k-\frac{n}{p}>0$.

<u>Note:</u> L^n_1 gauge transforms act on L^n connections.

When is $\mathcal{A}^p_{k-1}/\mathcal{G}^p_k = \mathcal{B}^p_{k-1}$ a Hausdorff space?

~~Fact~~ If G is a topological group and X is a Hausdorff topological space, then X/G is Hausdorff iff $\Gamma = \{(x, x\cdot g) \mid x \in X, g \in G\} \subset X \times X$ is closed.

<u>Claim:</u> If $(k-\frac{n}{p})>0$, then $\mathcal{B}^p_{k-1}$ is Hausdorff. Suppose we have (u_i, A_i) so that $(A_i, u_i A_i) \to (A, \tilde{A})$. Then $a^\tau_i \to a^\tau$ in L^p_{k-1}.

$u_i^{-1} du_i + u_i^{-1} a^\tau_i u_i =: \tilde{a}^\tau_i \to \tilde{a}^\tau \iff du_i = u_i \tilde{a}^\tau_i - a^\tau_i u_i$

$$\tilde{a}^\tau_i \xrightarrow{L^p_{k-1}} \tilde{a}^\tau, \qquad a^\tau_i \xrightarrow{L^p_{k-1}} a^\tau$$

Suppose $k=1$, $p=n+\varepsilon$.

$\Rightarrow du_i$ is uniformly bdd in $L^{n+\varepsilon}_1 \Rightarrow$ Pass to subseq where $u_i \xrightarrow{C^0} u$

$\Rightarrow du_i = u_i \tilde{a}^\tau_i - a^\tau_i u_i$ converge in $L^{n+\varepsilon} \Rightarrow u_i$ converge in $L^{n+\varepsilon}_1$ to u.

$\Rightarrow u^{-1}du + u^{-1}au$ in $L^{n+\varepsilon}$.