Review

Last time: for $U \subset \mathbb{R}^2$, we have

$$C^\infty(U,\mathbb{R}) \xrightarrow{\vec{\nabla}} C^\infty(U,\mathbb{R}^2) \xrightarrow{\text{rot}} C^\infty(U,\mathbb{R}) \longrightarrow 0\ , \tag{1}$$

and define

$$\begin{aligned} H^0(U) &:= \ker(\vec{\nabla}), \\ H^1(U) &:= \frac{\ker(\text{rot})}{\text{im}(\vec{\nabla})}, \\ H^2(U) &:= \frac{C^\infty(U,\mathbb{R})}{\text{im}(\text{rot})}. \end{aligned}$$

Lemma (Poincaré). *If $U \subset \mathbb{R}^2$ is star-shaped, then*

$$\begin{aligned} H^0(U) &\cong \mathbb{R}, \\ H^1(U) &\cong 0, \\ H^2(U) &\cong 0. \end{aligned}$$

Recall that $H^0(U)$ is the space of locally constant real-valued functions. Since all points of U are connected by a segment to some point $\vec{x}_0$, U consists of a single connected component. Therefore, locally constant functions on U are determined by a single constant in $\mathbb{R}$. $H^1(U) \cong 0$ is Theorem 1.4. $H^2(U) \cong 0$ was done in the handout.

Consequently, if U is star-shaped, then (1) is an exact sequence which resolves the subspace of constant functions in $C^\infty(U,\mathbb{R})$.

The story is very similar for $U \subset \mathbb{R}^3$. We have

$$C^\infty(U,\mathbb{R}) \xrightarrow{\vec{\nabla}} C^\infty(U,\mathbb{R}^3) \xrightarrow{\overrightarrow{\text{rot}}} C^\infty(U,\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(U,\mathbb{R}) \longrightarrow 0\ ,$$

and define

$$\begin{aligned} H^0(U) &:= \ker(\vec{\nabla}), \\ H^1(U) &:= \frac{\ker(\overrightarrow{\text{rot}})}{\text{im}(\vec{\nabla})}, \\ H^2(U) &:= \frac{\ker(\text{div})}{\text{im}(\overrightarrow{\text{rot}})}, \\ H^3(U) &:= \frac{C^\infty(U,\mathbb{R})}{\text{im}(\text{div})}. \end{aligned}$$

Lemma (Poincaré). *If $U \subset \mathbb{R}^3$ is star-shaped, then*

$$H^p(U) = \begin{cases} \mathbb{R} & \text{if } p = 0, \\ 0 & \text{if } p > 0. \end{cases}$$

The goal of the next several lectures will be to develop the tensor calculus necessary to generalize these results to $U \subset \mathbb{R}^n$. In particular, we will define tensors

$$C^\infty(U,\mathbb{R}) = \Omega^0(U) \xrightarrow{\vec{\nabla}=d^0} \Omega^1(U) \xrightarrow{d^1} \Omega^2(U) \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} \Omega^n(U) \xrightarrow{d^n} 0$$

which generalize the above cases $n = 2$ and $n = 3$. Then

$$H^0(U) := \ker d^0,$$
$$H^p(U) := \frac{\ker(\Omega^p \xrightarrow{d^p} \Omega^{p+1})}{\operatorname{im}(\Omega^{p-1} \xrightarrow{d^{p-1}} \Omega^p)}.$$

We will

- define $\Omega^p(U)$ and d^p,
- prove the generalized Poincaré lemma,
- develop tools to compute $H^p(U)$,
- ???
- PROFIT!!!

In the short-term,

- tensor algebra,
- tensor calculus,
- differential forms $\Omega^p(U)$ and d^p.

The tensor product

Reference: Hitchin's notes, Chapter 2

Suppose V and W are finite-dimensional real vector spaces. Define a vector space $V \otimes W$ according to the rules

- For every $v \in V$, $w \in W$ there is a product vector $v \otimes w \in V \otimes W$. (generators)
- Bilinearity (relations):

$$(\alpha v_1 + \beta v_2) \otimes w = \alpha v_1 \otimes w + \beta v_2 \otimes w, \tag{2}$$
$$v \otimes (\alpha w_1 + \beta w_2) = \alpha v \otimes w_1 + \beta v \otimes w_2.$$

A general element in $V \otimes W$ is a finite sum of the form $\sum_{k=1}^{\ell} \lambda_k \cdot v_k \otimes w_k$, where $0 \le \ell \in \mathbb{Z}$, and each $v_k \in V$ and $w_k \in W$.

Fact. *If* $\{e_i\}_{i=1}^m$ *is a basis for* V *and* $\{f_j\}_{j=1}^n$, *then* $\{e_i \otimes f_j\}$ *is a basis for* $V \otimes W$. *Therefore,* $\dim(V \otimes W) = \dim V \cdot \dim W$.

The proof will come later.

Example.

$$e_1 \otimes f_1 + 2e_1 \otimes f_2 + e_2 \otimes f_1 + 2e_2 \otimes f_2 \in V \otimes W.$$

An element of $V \otimes W$ is *decomposable* if it can be written as a single tensor product. For example, the above expression is

$$(e_1 + e_2) \otimes (f_1 + 2f_2).$$

Otherwise, a tensor is called *indecomposable*, for example

$$e_1 \otimes f_1 + e_2 \otimes f_2.$$

(This is not obvious.)

It's easy see that the basis vectors $\{e_i \otimes f_j\}$ span $V \otimes W$. This is because $V \otimes W$ is spanned by tensors of the form $v \otimes w$, and such tensors can be expanded as

$$v \otimes w = \left(\sum v_i e_i\right) \otimes \left(\sum w_j f_j\right) = \sum \sum \left(v_i w_j\right) \, e_i \otimes f_j.$$

Independence is harder, so we will defer this to the end.

Universal property

The tensor product is the universal bilinear operation. To understand what this means, consider the example of the cross product $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. The cross product is bilinear, so it is determined by the multiplication table

$$\begin{array}{lll} e_1 \times e_1 = 0 & e_1 \times e_2 = e_3 & e_1 \times e_3 = -e_2 \\ e_2 \times e_1 = -e_3 & e_2 \times e_2 = 0 & e_2 \times e_3 = e_1 \\ e_3 \times e_1 = e_2 & e_3 \times e_2 = -e_1 & e_3 \times e_3 = 0. \end{array}$$

The tensor product allows one to rewrite bilinear maps as linear maps. Specifically, we can define a *linear* map $\tilde{\times} : \mathbb{R}^3 \otimes \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\begin{array}{lll} \tilde{\times}(e_1 \otimes e_1) = 0 & \tilde{\times}(e_1 \otimes e_2) = e_3 & \tilde{\times}(e_1 \otimes e_3) = -e_2 \\ \tilde{\times}(e_2 \otimes e_1) = -e_3 & \tilde{\times}(e_2 \otimes e_2) = 0 & \tilde{\times}(e_2 \otimes e_3) = e_1 \\ \tilde{\times}(e_3 \otimes e_1) = e_2 & \tilde{\times}(e_3 \otimes e_2) = -e_1 & \tilde{\times}(e_3 \otimes e_3) = 0. \end{array}$$

Under this framework, it makes sense to compute

$$\tilde{\times}(e_1 \otimes e_2 + 5e_2 \otimes e_3) = \tilde{\times}(e_1 \otimes e_2) + 5\tilde{\times}(e_2 \otimes e_3) = (e_1 \times e_2) + 5(e_2 \times e_3) = e_3 + 5e_1.$$

More generally, if $B : V \times W \to U$ is any bilinear map, it makes sense to define $\tilde{B}(v \otimes w) = B(v, w)$ and extend linearly. Why is this well-defined? We need to verify that the value of $\tilde{B}(v \otimes w)$ does not depend on any particular expansion of $v \otimes w$. Specifically, we need to verify that

$$\tilde{B}\left((\alpha v_1 + \beta v_2) \otimes w\right) \stackrel{?}{=} \tilde{B}\left(\alpha v_1 \otimes w + \beta v_2 \otimes w\right).$$

Using the definition of $\tilde{B}$, and then using bilinearity, the left hand side is

$$B(\alpha v_1 + \beta v_2, w) = \alpha B(v_1, w) + \beta B(v_2, w).$$

Using linearity of $\tilde{B}$, and then using the definition of $\tilde{B}$, the right hand side is

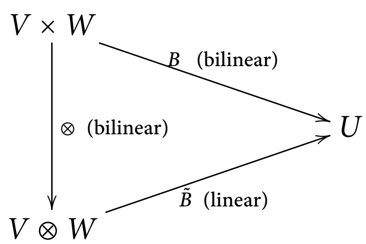
$$\alpha \tilde{B}(v_1 \otimes w) + \beta \tilde{B}(v_2 \otimes w) = \alpha B(v_1, w) + \beta B(v_2, w).$$

Thus our results agree. After also checking the second relation in (2), we conclude that $\tilde{B}$ is well-defined.

We are almost ready to state the universal property. First note that the tensor product itself is a bilinear map

$$\otimes : V \times W \to V \otimes W,$$
$$\otimes(v, w) := v \otimes w.$$

Universal property For any *bilinear* map $B : V \times W \to U$, there exists a unique *linear* map $\tilde{B} : V \otimes W \to U$ (as defined above) such that the following diagram commutes



i.e. $B = \tilde{B} \circ \otimes$.

Note This universal property characterizes the tensor product.

Example. Suppose $\{e_i\}_{i=1}^m$ and $\{f_j\}_{j=1}^n$ are bases of V and W. We define a bilinear map $M : V \times W \to \text{Mat}_{m\times n}$ to the space of $m \times n$ matrices by

$$M(v, w) = M(\sum v_i e_i, \sum w_j f_j) := [v_i w_j] = \begin{bmatrix} v_1 w_1 & v_1 w_2 & \cdots & v_1 w_n \\ v_2 w_1 & v_2 w_2 & \cdots & v_2 w_n \\ \vdots & \vdots & \ddots & \vdots \\ v_m w_1 & v_m w_2 & \cdots & v_m w_n \end{bmatrix}.$$

It's straightforward to check that M is bilinear, and that $M(e_i, f_j)$ is a matrix with a single 1 in the ij slot, and zeroes everywhere else.

We now finish our proof that $\{e_i \otimes f_j\}$ is a basis for $V \otimes W$ by showing that the $\{e_i \otimes f_j\}$ are independent.

Proof. Suppose there were a linear dependence

$$\sum a_{ij}\, e_i \otimes f_j = 0.$$

Then

$$\tilde{M}\left(\sum a_{ij}\, e_i \otimes f_j\right) = \tilde{M}(0) = \mathbf{0}.$$

On the other hand, we compute

$$\tilde{M}\left(\sum a_{ij}\, e_i \otimes f_j\right) = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Therefore we conclude

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \mathbf{0}.$$

Thus each $a_{ij} = 0$, so there is no nontrivial dependence. □

Multiple tensor products

Consider the triple product $T : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by

$$T(u, v, w) := (u \times v) \cdot w.$$

This is linear in each variable separately. Such a map with three arguments is called *trilinear*. For several arguments, the more general term is *multilinear*.

The multiplication table for T has 27 entries:

$$T(e_1, e_1, e_1) = 0, \;\cdots\; T(e_1, e_2, e_3) = 1, \;\cdots$$

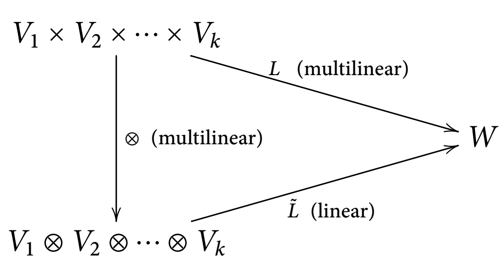
This table determines a linear map

$$\tilde{T}(e_1 \otimes e_1 \otimes e_1) = 0, \;\cdots\; \tilde{T}(e_1 \otimes e_2 \otimes e_3) = 1, \;\cdots$$

where

$$\tilde{T} : \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3 \to \mathbb{R}.$$

This is expressed using the generalized universal property that any multilinear map L corresponds to a unique linear map $\tilde{L}$:



Tensor product with $\mathbb{R}$

There is a simple bilinear map

$$m : \mathbb{R} \times V \longrightarrow V$$

defined by $m(\alpha, v) := \alpha v$. By the universal property, this is equivalent to a linear map

$$\tilde{m} : \mathbb{R} \otimes V \longrightarrow V$$

so that $\tilde{m}(\alpha \otimes v) = \alpha v$. This map has an inverse given by

$$\tilde{m}^{-1}(v) = 1 \otimes v.$$

The conclusion is that $\mathbb{R} \otimes V \cong V$ via the natural map $\tilde{m}$. Concretely, given a tensor in $\mathbb{R} \otimes V$, we can rewrite it, e.g.

$$3 \otimes e_1 - 5 \otimes e_2 = 3(1 \otimes e_1) - 5(1 \otimes e_2) = 1 \otimes (3e_1) + 1 \otimes (-5e_2) = 1 \otimes (3e_1 - 5e_2).$$

We use $\tilde{m}$ to identify this with $3e_1 - 5e_2 \in V$.

Tensor algebra

Let V be a vector space. Define the tensor power of a vector space V as follows:

$$\bigotimes^p V := \underbrace{V \otimes V \otimes \cdots \otimes V}_{p \text{ times}}.$$

It satisfies

$$(\bigotimes^p V) \otimes (\bigotimes^q V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{p \text{ times}} \otimes \underbrace{V \otimes V \otimes \cdots \otimes V}_{q \text{ times}} = \bigotimes^{p+q} V.$$

What is $\otimes^0 V$? It should satisfy

$$\left(\bigotimes^0 V\right) \otimes V = V,$$

so it makes sense to define $\otimes^0 V = \mathbb{R}$. We want to consider expressions like

$$\underbrace{4}_{\otimes^0 V} + \underbrace{5e_1 + e_2}_{\otimes^1 V} - \underbrace{e_1 \otimes e_3}_{\otimes^2 V} + \underbrace{2e_1 \otimes e_2 \otimes e_1}_{\otimes^3 V}.$$

The *tensor algebra* is defined to be

$$T(V) := \underbrace{\bigoplus_{p=0}^{\infty} \underbrace{\bigotimes^p V}_{\text{degree } p \text{ part}}}_{\substack{\text{finitely many nonzero} \\ \text{terms, allowing any} \\ \text{degree}}}$$

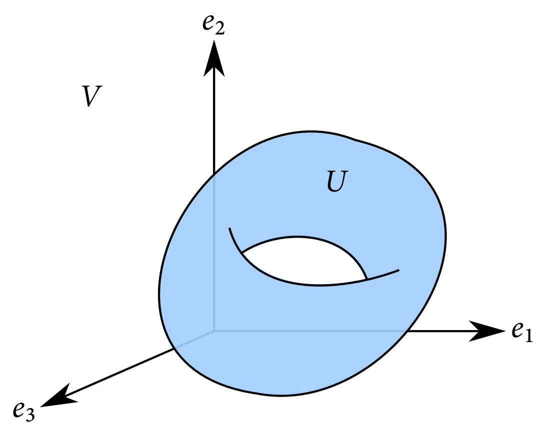
The word algebra refers to the fact that $T(V)$ is a vector space which is equipped with a multiplication (tensor product).

Example.

$$(2 + e_1) \otimes (e_3 + e_1 \otimes e_2) = 2e_3 + e_1 \otimes e_3 + 2e_1 \otimes e_2 + e_1 \otimes e_1 \otimes e_2.$$

Tensor calculus

Define $V := \mathbb{R}^n$. We have the standard basis $\{e_1, \ldots, e_n\}$ which correspond to the axis vectors. Correspondingly, we have the dual basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ which are the coordinate functions. Let $U \subset V$ be an open subset.



For $f \in C^\infty(U, \mathbb{R})$ we define the *total derivative*

$$Df := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \varepsilon_i \in C^\infty(U, V^*).$$

We call Df a *covector field* because it has values in the dual space.

Example. If $f(x_1, x_2) = x_1^3 x_2^4$, then

$$Df = 3x_1^2 x_2^4 \varepsilon_1 + 4x_1^3 x_2^3 \varepsilon_2.$$

We can evaluate Df at some point $x \in U$ to get an element of V^*, for example

$$Df|_{(x_1,x_2)=(1,1)} = 3\varepsilon_1 + 4\varepsilon_2 \in V^*$$

This covector encodes the partial derivatives into a directional derivative.

Recall that the *directional derivative* in the direction $v \in V$ is

$$(\partial_{\vec{v}} f)(\vec{x}) := \lim_{h \to 0} \frac{f(\vec{x} + h\vec{v}) - f(\vec{x})}{h}.$$

For $f \in C^1(U)$ (and therefore for $f \in C^\infty$), if $v = \sum v_i e_i$, then by the chain rule,

$$\partial_v f = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}.$$

In our example, if we want to compute the directional derivative at the point $(1, 1)$ in the direction of $v = 2e_1 + e_2$, we evaluate

$$Df|_{(1,1)}(v) = (3\varepsilon_1 + 4\varepsilon_2)(2e_1 + e_2) = 3 \cdot 2 + 4 \cdot 1 = 10.$$

Higher derivatives

Let W be any vector space, and consider $\phi \in C^\infty(U, W)$. Taking a basis $\{w_j\}$ of W, we can write ϕ in components as

$$\phi = \sum \phi_j w_j$$

where $\phi_j \in C^\infty(U, W)$. We extend our definition of total derivative by

$$D\phi := \sum (D\phi_j) \otimes w_j \in C^\infty(U, V^* \otimes W).$$

Often W will itself be a tensor product or tensor algebra. For example, iterating D we get the following chain of maps:

$$C^\infty(U, \mathbb{R}) \overset{D}{\to} C^\infty(U, V^*) \overset{D}{\to} C^\infty(U, V^* \otimes V^*) \overset{D}{\to} C^\infty(U, \otimes^3 V^*) \overset{D}{\to} \cdots$$

In particular, D acts on the tensor algebra, increasing degrees by one:

$$D : C^\infty(U, T(V^*)) \to C^\infty(U, T(V^*)).$$

For example,

$$D(x_1^2 \varepsilon_3 \otimes \varepsilon_2 + x_2) = 2x_1 \varepsilon_1 \otimes \varepsilon_3 \otimes \varepsilon_2 + \varepsilon_2.$$

Continuing with the previous example of $f(x_1, x_2) = x_1^3 x_2^4$ with $Df = 3x_1^2 x_2^4 \varepsilon_1 + 4x_1^3 x_2^3 \varepsilon_2$, we compute

$$\begin{aligned} D^2 f = D(Df) &= D(3x_1^2 x_2^4) \otimes \varepsilon_1 + D(4x_1^3 x_2^3) \otimes \varepsilon_2 = \\ &= (6x_1 x_2^4 \varepsilon_1 + 12x_1^2 x_2^3 \varepsilon_2) \otimes \varepsilon_1 + (12x_1^2 x_2^3 \varepsilon_1 + 12x_1^3 x_2^2 \varepsilon_2) \otimes \varepsilon_2 \\ &= 6x_1 x_2^4 \varepsilon_1 \otimes \varepsilon_1 + 12x_1^2 x_2^3 (\varepsilon_1 \otimes \varepsilon_2 + \varepsilon_2 \otimes \varepsilon_1) + 12x_1^3 x_2^2 \varepsilon_2 \otimes \varepsilon_2. \end{aligned}$$

The total derivatives encode the partial derivatives. To extract partial derivatives from these tensors, we need a notion of evaluation.

Evaluation of tensors I

Suppose $t \in V \otimes W$. We can view t as a bilinear map $t : V^* \times W^* \to \mathbb{R}$ according to

$$v \otimes w(\alpha, \beta) := \alpha(v) \cdot \beta(w).$$

(When t is not decomposable, we use this formula for each term.) For example, if $t = 5e_1 \otimes e_2 + 3e_2 \otimes e_1$, then we can perform the evaluation

$$\begin{aligned} t(2\varepsilon_1, \varepsilon_1 + \varepsilon_2) &= 5 \cdot 2\varepsilon_1(e_1) \cdot (\varepsilon_1 + \varepsilon_2)(e_2) + 3 \cdot 2\varepsilon_1(e_2) \cdot (\varepsilon_1 + \varepsilon_2)(e_1) \\ &= 5 \cdot 2 \cdot 1 \cdot (0 + 1) + 3 \cdot 2 \cdot 0 \cdot (1 + 0) = 10. \end{aligned}$$

This isn't quite what we want, since we wish to evaluate the tensor $D^2 f$, which takes values in $V^* \otimes V^*$.

Dual spaces and double-duals

Given a finite-dimensional real vector space V, the dual space V^* is defined as

$$V^* := \{\alpha : V \to \mathbb{R} \mid \alpha \text{ is linear}\}.$$

Given a basis $\{e_i\}_{i=1}^n$ of V, there is a standard way to construct a basis of V^*, called the *dual basis* $\{\varepsilon_j\}_{j=1}^n$. This basis is defined by

$$\varepsilon_j(e_i) := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and extending the ε_j linearly. Defined this way, the ε_j are coordinate functions. For instance, if

$$v = v_1 e_1 + \cdots + v_n e_n,$$

then the components of v are

$$(v_1, \ldots, v_n).$$

The j-th coordinate of v is then

$$\varepsilon_j(v) = \varepsilon_j(v_1 e_1 + \cdots + v_n e_n) = v_1 \varepsilon_j(e_1) + \cdots + v_j \varepsilon_j(e_j) + \cdots + v_n \varepsilon_j(e_n) = 0 + \cdots + 0 + v_j \cdot 1 + 0 + \cdots + 0 = v_j.$$

Why do we care about dual spaces? If V is an inner product space and $\{e_i\}$ is an orthonormal basis, then coordinate functions are given by the dot product

$$v_i = e_i \cdot v.$$

But this requires V to have some fixed inner product. Furthermore it requires us to choose an orthonormal basis. If we wanted to study Riemannian geometry we would do this, but keeping track of inner products quickly gets complicated. Keeping track of dual spaces instead will save work in the long run.

Note that $\dim V = \dim V^*$, and so abstractly, $V \cong V^*$. For instance, we could construct an isomorphism $V \to V^*$ by sending $e_i \mapsto \varepsilon_i$. However, the resulting isomorphism depends on the choice of basis!!! Just as there is no natural choice of basis for an abstract vector space V, there is no natural isomorphism between V and V^*. In order to *identify* V^* with V, we would need a *particular* isomorphism $V \cong V^*$. Since there is no way to choose a distinguished isomorphism, we must think of V and V^* as distinct vector spaces.

The situation is different for V^{**}. There is an evaluation isomorphism

$$\begin{aligned} \mathrm{ev} : V &\to V^{**}, \\ \underbrace{v}_{V} &\mapsto \underbrace{\big(\underbrace{\alpha}_{V^*} \mapsto \underbrace{\alpha(v)}_{\mathbb{R}}\big)}_{V^{**}}. \end{aligned}$$

To unpack this definition, let's first note that $V^{**} = \{\text{linear maps } V^* \to \mathbb{R}\}$. Fix some $v \in V$. Then $\mathrm{ev}(v) \in V^{**}$. To improve notation, let's write ev_v for $\mathrm{ev}(v)$. Thus $\mathrm{ev}_v : V^* \to \mathbb{R}$. Suppose $\alpha \in V^*$. We need to specify $\mathrm{ev}_v(\alpha)$. How do we get a real number from v and α?

$$\mathrm{ev}_v(\alpha) := \alpha(v),$$

so we evaluate α on v. It's easy to check that $\mathrm{ev}_v : V^* \to \mathbb{R}$ is linear:

$$\mathrm{ev}_v(2\alpha + 3\beta) = (2\alpha + 3\beta)(v) = 2\alpha(v) + 3\beta(v) = 2\mathrm{ev}_v(\alpha) + 3\mathrm{ev}_v(\beta).$$

Thus $\mathrm{ev}_v \in V^{**}$.

Theorem. *The map* $\mathrm{ev} : V \to V^{**}$ *given by* $v \mapsto \mathrm{ev}_v$ *is an isomorphism.*

Proof. Suppose $v \in \ker(\mathrm{ev})$. Then

$$(\alpha \mapsto \alpha(v)) = \underbrace{(\alpha \mapsto 0)}_{0 \in V^*}.$$

$$\implies \alpha(v) = 0 \text{ for all } \alpha$$

$$\implies \text{all components of } v \text{ are zero}$$

$$\implies v = 0.$$

We also need to show that ev is surjective. This now follows from the fact that V and V^{**} have the same dimension:

$$\dim(\mathrm{image}(\mathrm{ev})) = \dim(V) - \dim(\ker(\mathrm{ev})) = \dim V = \dim V^* = \dim V^{**},$$

so $\mathrm{image}(\mathrm{ev}) = V^{**}$. □

While $V^* \neq V$, even though they are isomorphic, we can consider $V^{**} = V$ by using ev as our understood identification.

We can view this identification as "symmetry of the dual pairing" as follows. For any vector space W, define the *dual pairing* $\langle \bullet, \bullet \rangle_W : W^* \times W \to \mathbb{R}$ by

$$\langle \beta, w \rangle_W := \beta(w).$$

Note that this does not use an inner product on W since $\beta \in W^*$, and $\langle w_1, w_2 \rangle_W$ is undefined.

For $v \in V$ and $\alpha \in V^*$, observe that

$$\langle \alpha, v \rangle_V = \alpha(v) = \mathrm{ev}_v(\alpha) = \langle \mathrm{ev}_v, \alpha \rangle_{V^*}.$$

Using the identification $v \leftrightarrow \mathrm{ev}_v$, we may write

$$\langle \alpha, v \rangle_V = \langle v, \alpha \rangle_{V^*},$$

so the dual pairing, although not an inner product, is still symmetric!

Evaluation of tensors II

According to our previous formula, we can view $t \in V \otimes W$ as a bilinear map $t : V^* \times W^* \to \mathbb{R}$ according to

$$v \otimes w(\alpha, \beta) = \langle \alpha, v \rangle_V \langle \beta, w \rangle_W .$$

Using our identifications, we may view $t \in V^* \otimes W^*$, as a bilinear map $t : V \times W \to \mathbb{R}$ defined by

$$\alpha \otimes \beta(v, w) := \langle v, \alpha \rangle_{V^*} \langle w, \beta \rangle_{W^*} = \langle \alpha, v \rangle_V \langle \beta, w \rangle_{W^*} .$$

For example, if $f(x_1, x_2) = x_1^3 x_2^4$, then

$$D^2 f = 6x_1 x_2^4 \varepsilon_1 \otimes \varepsilon_1 + 12x_1^2 x_2^3 (\varepsilon_1 \otimes \varepsilon_2 + \varepsilon_2 \otimes \varepsilon_1) + 12x_1^3 x_2^2 \varepsilon_2 \otimes \varepsilon_2,$$

and

$$D^2 f(e_2, e_1) = 6x_1 x_2^4 \cdot 0 \cdot 1 + 12x_1^2 x_2^3 (0 \cdot 0 + 1 \cdot 1) + 12x_1^3 x_2^2 \cdot 1 \cdot 0 = 12x_1^2 x_2^3 = \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_1} f.$$

More generally, if we let ∂_v denote the directional derivative along the vector v, then

$$D^2 f(v_1, v_2) = \partial_{v_1} \partial_{v_2} f.$$

Symmetry of D^2

Our goal is to turn

$$C^\infty(U, \mathbb{R}) \xrightarrow{D} C^\infty(U, V^*) \xrightarrow{D} C^\infty(U, V^* \otimes V^*) \xrightarrow{D} C^\infty(U, \otimes^3 V^*) \xrightarrow{D} \cdots$$

into a resolution, however, $D^2 f \neq 0$. We will eventually fix this.

For a general function $f \in C^\infty(U, \mathbb{R})$,

$$D^2 f = D\left(\sum_j \frac{\partial f}{\partial x_j} \varepsilon_j \right) = \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \varepsilon_i \otimes \varepsilon_j \in C^\infty(U, V^* \otimes V^*).$$

Theorem. *If $f \in C^2(U, \mathbb{R})$ (and hence if $f \in C^\infty$), then $\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f = \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} f$.*

The proof is surprisingly subtle. Spivak describes a proof which uses Fubini's Theorem. Alternatively, one could provide a straightforward proof using Taylor polynomials. (To show equality at a given point, perform the second-order Taylor expansion. Then simple computations show both that partial derivatives are symmetric on polynomials, and that the remainder does not contribute.)

Corollary. *$D^2 f$ is a symmetric tensor.*

A tensor $t \in W \otimes W$ is *symmetric* if one of the following equivalent conditions hold.

- $t(\alpha, \beta) = t(\beta, \alpha)$ for all $\alpha, \beta \in W^*$.

- The components $\left[t_{ij}\right]$ form a symmetric matrix, i.e. $t_{ij} = t_{ji}$.
- $\sigma(t) = t$, where $\sigma : W \otimes W \to W \otimes W$ is the swap map generated by $\sigma(w_1 \otimes w_2) = w_2 \otimes w_1$.

Equivalence of these conditions is simple to verify. There is one remaining characterization which is slightly trickier, but it will lead to our construction of the exterior algebra.

Theorem. *A tensor $t \in W \otimes W$ is symmetric, iff t is in the span of $\{w \otimes w \mid w \in W\}$.*

Proof. If t is in the span of $\{w \otimes w \mid w \in W\}$, then $\sigma(t) = t$, so t is symmetric. Conversely, we must show that if t is symmetric, then it is spanned by vectors of the form $w \otimes w$. For this, we will use the polarization identity

$$w_1 \otimes w_2 + w_2 \otimes w_1 = (w_1 + w_2) \otimes (w_1 + w_2) - w_1 \otimes w_1 - w_2 \otimes w_2.$$

This identity explains how to rewrite something in the form of the left hand side as terms of the form $w \otimes w$.

Suppose $\{f_j\}$ is a basis for W. Then

$$t = \sum_{i,j} t_{ij}\, f_i \otimes f_j = \sum_{i,j} t_{ji}\, f_i \otimes f_j = \sum_{j,i} t_{ij}\, f_j \otimes f_i,$$

where for the second equality we used the symmetry $t_{ij} = t_{ji}$, and for the last equality we exchanged the names of the dummy variables i and j. Thus

$$\begin{aligned} t &= \tfrac{1}{2}t + \tfrac{1}{2}t = \tfrac{1}{2}\sum_{i,j} t_{ij}\, f_i \otimes f_j + \tfrac{1}{2}\sum_{j,i} t_{ij}\, f_j \otimes f_i = \sum_{i,j} \tfrac{1}{2} t_{ij}(f_i \otimes f_j + f_j \otimes f_i) \\ &= \sum_{i,j} \tfrac{1}{2} t_{ij} \left((f_i + f_j) \otimes (f_i + f_j) - f_i \otimes f_i - f_j \otimes f_j \right) \in \operatorname{span}\{w \otimes w \mid w \in W\}. \end{aligned}$$

□

Definition of the exterior algebra and differential forms

To impose $D^2 f = 0$, we kill the symmetric part by taking the quotient

$$V^* \wedge V^* := \frac{V^* \otimes V^*}{\operatorname{span}\{\alpha \otimes \alpha \mid \alpha \in V^*\}}.$$

More generally, we will take a quotient of the tensor algebra $T(V^*)$. Let I denote the ideal in $T(V^*)$ generated by elements of the form $\alpha \otimes \alpha$ with $\alpha \in V^*$, i.e. the additive *and multiplicative* span. For instance,

$$\alpha_1 \otimes \underline{\alpha_2 \otimes \alpha_2} \otimes \alpha_3 + \underline{\alpha_4 \otimes \alpha_4} \otimes \alpha_5 \in I.$$

Definition. The *exterior algebra* is defined as

$$\Lambda^\bullet V^* := \frac{T(V^*)}{I}.$$

The product in the exterior algebra is written as the *wedge product* $\wedge$ instead of $\otimes$.

Taking the quotient by I has the effect of literally setting $\alpha \wedge \alpha = 0$ for any $\alpha \in V^*$.

Definition. The p-th exterior power of V^* is

$$\Lambda^p V^* := \{\omega \in \Lambda^\bullet V^* \mid \omega \text{ is homogeneous of degree } p\}.$$

The degree of ω refers to the number of V^* factors.

Example.

$$5\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 + \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4 \in \Lambda^3 V^*.$$
$$5\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 + \varepsilon_2 \wedge \varepsilon_3 \notin \Lambda^3 V^*.$$

The second expression is not in $\Lambda^3 V^*$ because it's not homogeneous: the second term has degree two.

Definition. *Differential forms* are smooth functions with values in the exterior algebra:

$$\Omega^\bullet(U) := C^\infty(U, \Lambda^\bullet V^*),$$
$$\Omega^p(U) := C^\infty(U, \Lambda^p V^*).$$

In other words, the coefficients of $\Lambda^\bullet V^*$ are functions on U rather than scalars.

Example.

$$x_2^4 x_3^5\, \varepsilon_1 \wedge \varepsilon_2 + \frac{\varepsilon_2 \wedge \varepsilon_3}{x_1^2 + x_2^2 + x_3^2} \in \Omega^2(\mathbb{R}^3 - \{\mathbf{0}\}).$$

Definition. The *exterior derivative* $d : \Omega^\bullet(U) \to \Omega^\bullet(U)$ is the operator on $\Omega^\bullet(U)$ induced by the total derivative D. Similarly, $\Omega^p(U) := C^\infty(U, \Lambda^p V^*)$.

Since taking a derivative adds one factor to any tensor, $d : \Omega^p(U) \to \Omega^{p+1}(U)$.

Example.

$$d(x_1^3\, \varepsilon_2 \wedge \varepsilon_3) = 3x_1^2\, \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3.$$

Note Most people write dx_i for ε_i since

$$d(x_i) = \varepsilon_i.$$

Properties of the exterior algebra

Here are the important properties of the exterior algebra:

1. If $\dim V = n$, then a basis for $\Lambda^p V^*$ is given by

$$\left\{\varepsilon_{I_1} \wedge \varepsilon_{I_2} \wedge \cdots \wedge \varepsilon_{I_p} \mid I \text{ is a (strictly increasing) subsequence of } \{1, \ldots, n\} \text{ with length } p\right\}. \tag{3}$$

   For shorthand, we write

$$\varepsilon_I := \varepsilon_{I_1} \wedge \cdots \wedge \varepsilon_{I_p},$$

   so that our basis is $\{\varepsilon_I\}_{I \text{ strictly increasing of length } p}$. Increasing sequences of length p correspond to subsets of size p from $\{1, \ldots, n\}$. Consequently, $\dim \Lambda^p V^* = \binom{n}{p}$. In particular, $\Lambda^p V^* = 0$ if $p > n$.

$\dim \Lambda^p V^*$ $p =$	0	1	2	3	4	5
$n = 0$	1	0	0	0	0	0
$n = 1$	1	1	0	0	0	0
$n = 2$	1	2	1	0	0	0
$n = 3$	1	3	3	1	0	0
$n = 4$	1	4	6	4	1	0
$n = 5$	1	5	10	10	5	1

2. If $\alpha, \beta \in V^* = \Lambda^1 V^*$, then
$$\alpha \wedge \beta = -\beta \wedge \alpha.$$

3. (Hitchin's Proposition 5.2) If $\alpha \in \Lambda^p V^*$ and $\beta \in \Lambda^q V^*$, then $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$.

4. If $\alpha = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_p$ with each $\alpha_k \in V^*$, then transposing α_i and α_j results in a minus sign, i.e.
$$\begin{aligned} \alpha &= \ \ \alpha_1 \wedge \cdots \wedge \alpha_{i-1} \wedge \underline{\alpha_i} \wedge \alpha_{i+1} \wedge \cdots \wedge \alpha_{j-1} \wedge \underline{\alpha_j} \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_p \\ &= -\alpha_1 \wedge \cdots \wedge \alpha_{i-1} \wedge \alpha_j \wedge \alpha_{i+1} \wedge \cdots \wedge \alpha_{j-1} \wedge \alpha_i \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_p. \end{aligned}$$

5. If $\sigma \in \mathcal{S}_p$ is a permutation of $\{1, \ldots, p\}$, then
$$\alpha = \operatorname{sign}(\sigma)\, \alpha_{\sigma(1)} \wedge \cdots \wedge \alpha_{\sigma(p)}.$$

6. There is a bijective correspondence between $\Lambda^p V^*$ and the space $\mathrm{Alt}^p(V)$ of alternating multilinear maps $\underbrace{V \times \cdots \times V}_{p} \to \mathbb{R}$. (Such a multilinear map A is called *alternating* if $A(v_1, \ldots, v_p) = 0$ whenever $v_i = v_j$ for $i \neq j$). For a decomposable $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_p \in \Lambda^p V^*$, the corresponding alternating multilinear map is the "evaluation"
$$\alpha(v_1, v_2, \ldots, v_p) := \sum_{\sigma \in \mathcal{S}_p} \operatorname{sign}(\sigma) \cdot \alpha_{\sigma(1)}(v_1) \cdot \alpha_{\sigma(2)}(v_2) \cdots \alpha_{\sigma(p)}(v_p).$$

The exterior algebra is just like the tensor algebra, except that $\alpha \wedge \alpha = 0$ whenever $\alpha \in V^*$. To prove these properties, we need to understand the consequences of this relation.

Example.
$$\varepsilon_1 \wedge \cancel{\varepsilon_2 \wedge \varepsilon_2} \wedge \varepsilon_3 = 0.$$

Example. If $\alpha, \beta \in V^*$, then
$$0 = (\alpha + \beta) \wedge (\alpha + \beta) = \cancel{\alpha \wedge \alpha} + \alpha \wedge \beta + \beta \wedge \alpha + \cancel{\beta \wedge \beta} \implies \alpha \wedge \beta = -\beta \wedge \alpha.$$

This proves Property 2. It's useful for reordering vectors in a wedge product.

Example.

$$\begin{aligned}&\underline{\varepsilon_1 \wedge \varepsilon_2} \wedge \varepsilon_3 + \underline{\varepsilon_3 \wedge \varepsilon_2} \wedge \varepsilon_1 \\ =&- \varepsilon_2 \wedge \underline{\varepsilon_1 \wedge \varepsilon_3} - \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_1 \\ =&+ \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_1 - \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_1 \\ =&0.\end{aligned}$$

Example. If $\dim V = 4$, then according to Property 1,

$$\{\varepsilon_1 \wedge \varepsilon_2,\ \varepsilon_1 \wedge \varepsilon_3,\ \varepsilon_1 \wedge \varepsilon_4,\ \varepsilon_2 \wedge \varepsilon_3,\ \varepsilon_2 \wedge \varepsilon_4,\ \varepsilon_3 \wedge \varepsilon_4\}$$

is a basis of $\Lambda^2 V^*$.

It's easy to show that the basis vectors (3) span $\Lambda^p V^*$, but it is difficult to prove that they are linearly independent. Here is the proof that they span.

Proof. Since $\Lambda^p V^*$ is spanned by decomposables, it suffices to show that (3) spans any decomposable. Suppose $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_p$ is such a decomposable. We can expand each α_i in terms of the ε_i according to

$$\alpha_i = \sum_j a_{ij}\, \varepsilon_j$$

for some collection of scalars $a_{ij} \in \mathbb{R}$. Specifically,

$$\begin{aligned}\alpha &= (a_{11}\varepsilon_1 + a_{12}\varepsilon_2 + \cdots + a_{1n}\varepsilon_n) \wedge (a_{21}\varepsilon_1 + \cdots) \wedge \cdots \wedge (a_{p1}\varepsilon_1 + a_{p2}\varepsilon_2 + \cdots + a_{pn}\varepsilon_n) \\ &= a_{11}a_{21}\cdots a_{p1}\, \varepsilon_1 \wedge \varepsilon_1 \wedge \cdots \wedge \varepsilon_1 + a_{11}a_{21}\cdots a_{p2}\, \varepsilon_1 \wedge \varepsilon_1 \wedge \cdots \wedge \varepsilon_2 + \cdots \\ &\in \operatorname{span}\left\{\varepsilon_{I_1} \wedge \varepsilon_{I_2} \wedge \cdots \wedge \varepsilon_{I_p} \mid I \text{ is a sequence of numbers from } \{1, \ldots, n\} \text{ of length } p\right\}.\end{aligned}$$

We are free to swap adjacent elements, since the minus sign does not affect the span. Hence we can reorder each product so that the sequence is ascending. Sequences containing a duplicate correspond to products containing some $\varepsilon_j \wedge \varepsilon_j$, so they vanish. All that remain are the strictly increasing sequences. □

Property 3 follows from iterating the swap operation.

Proof. Suppose α, β are decomposable so that $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_p$ and $\beta = \beta_1 \wedge \cdots \wedge \beta_q$ with each $\alpha_i \in V^*$ and $\beta_j \in V^*$. Then

$$\begin{aligned}\alpha \wedge \beta &= \alpha_1 \wedge \cdots \wedge \alpha_{p-1} \wedge \underline{\alpha_p \wedge \beta_1} \wedge \beta_2 \wedge \cdots \wedge \beta_q \\ &= -\alpha_1 \wedge \cdots \wedge \alpha_{p-2} \wedge \underline{\alpha_{p-1} \wedge \beta_1} \wedge \alpha_p \wedge \beta_2 \wedge \beta_3 \wedge \cdots \wedge \beta_q \\ &= (-1)^2 \alpha_1 \wedge \cdots \wedge \underline{\alpha_{p-2} \wedge \beta_1} \wedge \alpha_{p-1} \wedge \alpha_p \wedge \beta_2 \wedge \beta_3 \wedge \cdots \wedge \beta_q \\ &\ \ \vdots \\ &= (-1)^p \beta_1 \wedge \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_p \wedge \beta_2 \wedge \beta_3 \wedge \cdots \wedge \beta_q \\ &= (-1)^{2p} \beta_1 \wedge \beta_2 \wedge \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_p \wedge \beta_3 \wedge \beta_4 \wedge \cdots \wedge \beta_q \\ &= (-1)^{pq} \beta_1 \wedge \cdots \wedge \beta_q \wedge \alpha_1 \wedge \cdots \wedge \alpha_p \\ &= (-1)^{pq} \beta \wedge \alpha.\end{aligned}$$

In general, α and β are each sums of such decomposable terms. Expanding the product, each cross term acquires a common factor of $(-1)^{pq}$ when the order is interchanged. The result factors as $(-1)^{pq}\beta \wedge \alpha$. □

The proof of Property 4 is best illustrated by example.

Example. Suppose we wish to exchange α_2 and α_6 in

$$\alpha = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6 \wedge \alpha_7.$$

First we move α_6 past α_3, α_4 and α_5. Then

$$\alpha = (-1)^3 \alpha_1 \wedge \alpha_2 \wedge \alpha_6 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_7.$$

To obtain the desired result, we move α_2 past the same covectors α_3, α_4 and α_5, and also α_6. Thus

$$\alpha = (-1)^3(-1)^{3+1} \alpha_1 \wedge \alpha_6 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_2 \wedge \alpha_7.$$

More generally, the total number of swaps required to exchange α_i and α_j is one more than twice the number of factors between α_i and α_j, and hence is always odd.

We need some group theory for Property 5.

Definition. Given a permutation $\sigma \in \mathcal{S}_p$, the *permutation matrix* is the matrix M_σ determined by the linear transformation of $\mathbb{R}^p$ determined by sending the basis vectors $f_i \mapsto f_{\sigma(i)}$.

Example. If $\sigma \in \mathcal{S}_3$ is the permutation

$$\begin{aligned} \sigma(1) &= 1, \\ \sigma(2) &= 3, \\ \sigma(3) &= 2, \end{aligned}$$

then

$$M_\sigma = \begin{bmatrix} | & | & | \\ f_1 & f_3 & f_2 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that $\sigma \mapsto M_\sigma$ is a group homomorphism from the permutation group $\mathcal{S}_p$ to the group of $p \times p$ invertible matrices, i.e.

$$\sigma_1 \circ \sigma_2 = M_{\sigma_1} M_{\sigma_2}.$$

Definition. The *sign* of a permutation σ is

$$\operatorname{sign}(\sigma) := \det M_\sigma.$$

Definition. A *transposition* is a permutation which swaps two elements, leaving all others fixed.

Theorem. *If $\tau \in \mathcal{S}_p$ is a transposition, then* $\operatorname{sign}(\tau) = -1$.

Proof. M_τ is the identity matrix with two columns swapped. Swapping columns changes the determinant by a factor of -1. Thus $\det M_\tau = -\det(\mathrm{Id}_{p\times p}) = -1$. □

It's intuitively obvious that the group $\mathcal{S}_p$ is generated by transpositions. Specifically, we can rearrange p objects in any order by performing only swaps. Mathematically, any element $\sigma \in \mathcal{S}_p$ can be written (not uniquely!) as $\sigma = \tau_1\tau_2\cdots\tau_k$ for some collection of k transpositions. The number k depends on σ in a very specific way.

Corollary. *If $\sigma = \tau_1\tau_2\cdots\tau_k$ where the τ_k are transpositions, then* $\mathrm{sign}(\sigma) = (-1)^k$.

Proof.

$$\mathrm{sign}(\sigma) = \det M_\sigma = \det M_{\tau_1\cdots\tau_k} = (\det M_{\tau_1})\cdots(\det M_{\tau_k}) = (-1)^k.$$

□

Now we can prove Property 5. Given $\sigma \in \mathcal{S}_p$, write $\sigma = \tau_1\cdots\tau_k$. Then

$$\alpha = \alpha_1 \wedge \cdots \wedge \alpha_p$$

can be rearranged by k transpositions to

$$\alpha = (-1)^k \alpha_{\sigma(1)} \wedge \cdots \wedge \alpha_{\sigma(p)} = \mathrm{sign}(\sigma)\alpha_{\sigma(1)} \wedge \cdots \wedge \alpha_{\sigma(p)}.$$

Relation to physics

Exterior algebras occur in physics when studying fermions. Suppose the number of pure states is finite. For example, a single particle of spin s can occupy $(2s+1)$ possible pure states. Pure states of a single particle correspond to basis elements $\{\varepsilon_i\}$ of some (complex) vector space, say V^*. A collection of p identical fermions corresponds to an element of $\Lambda^p V^*$. Identical particles can be classified as either bosons or fermions. Fermions anticommute with each other, while bosons commute with both bosons and fermions. Specifically, when two fermions are transposed, their collective wavefunction acquires a factor of -1. In Property 3, the factor of $(-1)^{pq}$ can be interpreted as saying that an even number of fermions behaves as a boson, while an odd number of fermions behaves as a fermion:

$$\Lambda^{\text{even}} V^* \text{ are bosons.}$$
$$\Lambda^{\text{odd}} V^* \text{ are fermions,}$$

One consequence is the Pauli exclusion principle, that no two fermions can occupy the same state. Concretely, any basis element ε_I contains no duplicates.

Supersymmetry is a symmetry which exchanges bosons and fermions. The de Rham complex is supersymmetric. An explanation of supersymmetry is far beyond the scope of this course, but one consequence is that

$$\dim \Lambda^{\text{odd}} V^* = \dim \Lambda^{\text{even}} V^*.$$

For example, in the case $n = \dim V = 4$, the dimensions of $\Lambda^p V^*$ are so

$$\dim \Lambda^{\text{even}} V^* = 1 + 6 + 1 = 8,$$
$$\dim \Lambda^{\text{odd}} V^* = 4 + 4 = 8.$$

$p =$	0	1	2	3	4	5
$\dim \Lambda^p V^*$	1	4	6	4	1	0

Evaluation of wedge products

We have not yet proven Property 1. In particular, we have not yet shown that there are nonzero elements of $\Lambda^p V^*$! Property 6 is analogous to the universal property of tensor products. Since there exist nonzero alternating maps, we conclude that there exist nonzero elements of $\Lambda^p V^*$.

We need to check that our definition in Property 6 of the evaulation map makes sense. We will tackle this issue in the next section, first we consider some concrete examples.

Example. For $p = 2$, the group $\mathcal{S}_2$ has two elements, and

$$\varepsilon_2 \wedge \varepsilon_3(v, w) = \varepsilon_2(v)\varepsilon_3(w) - \varepsilon_3(v)\varepsilon_2(w) = v_2 w_3 - v_3 w_2.$$

This is

- the cross product of the projections of v and w to the x_2-x_3 plane,
- the signed area of the parallelogram in $\mathbb{R}^2$ with sides given by the projections of v and w,
- the determinant of the matrix $\begin{bmatrix} v_2 & w_2 \\ v_3 & w_3 \end{bmatrix}$.

Example. Moving up, for $p = 3$, the group $\mathcal{S}_3$ has six elements, and

$$\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3(u, v, w) = u_1 v_2 w_3 - u_1 v_3 w_2 - u_2 v_1 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_3 v_2 w_1.$$

This is

- the triple product $(u \times v) \cdot w$,
- the signed area of the parallelepiped with sides u, v, w,
- the determinant of $\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$.

Example. Suppose I, J are strictly increasing sequences of length p. Then

$$\varepsilon_I(e_J) = \varepsilon_{I_1} \wedge \cdots \wedge \varepsilon_{I_p}(e_{J_1}, \dots, e_{J_p}) = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{otherwise,} \end{cases}$$

and only the term corresponding to the identity permutation contributes the 1. This example shows that $\{e_I\}_{\substack{I \text{ strictly increasing} \\ \text{of length } p}}$ serves as a dual basis for $\Lambda^p V^*$. Evaluation on e_J produces the ε_I component, so just as

$$v = \sum_{i=1}^{n} \varepsilon_i(v)\, e_i \text{ for } v \in V,$$

we have

$$\omega = \sum_{\substack{I \text{ strictly increasing} \\ \text{of length } p}} \omega(e_I)\, \varepsilon_I.$$

Why evaluation is well-defined

Unless one is using a fixed basis, there are many possible ways to express any given element of $\Lambda^p V^*$. For example, if $\omega = \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4$, then we can rewrite this as

$$\omega = \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4 + \varepsilon_1 \wedge (\varepsilon_2 + 2\varepsilon_3) \wedge (\varepsilon_2 + 2\varepsilon_3).$$

Both expressions are different representatives of the same element. In order to show that our definition makes sense, we must show that we obtain the same alternating map, regardless of our particular representative.

$$\begin{aligned} & \text{different representatives give the same map} \\ \iff & \text{difference between two representatives gives the zero map} \\ \iff & \text{the ideal } I \text{ generated by } \alpha \wedge \alpha \text{ for } \alpha \in V^* \text{ gives the zero map} \\ \iff & \alpha_1 \wedge \cdots \wedge \alpha_p \text{ maps to zero whenever } \alpha_k = \alpha_{k+1} \text{ for some } k. \end{aligned}$$

Although it is obvious that $\alpha_1 \wedge \cdots \wedge \alpha_p = 0$ whenever $\alpha_k = \alpha_{k+1}$ for some k, so that $\alpha_1 \wedge \cdots \wedge \alpha_p(v_1, \ldots, v_p)$ ought to be zero, it's not obvious from our definition that

$$\sum_{\sigma \in \mathcal{S}_p} \operatorname{sign}(\sigma) \cdot \alpha_{\sigma(1)}(v_1) \cdot \alpha_{\sigma(2)}(v_2) \cdots \alpha_{\sigma(p)}(v_p) = 0 \text{ whenever } \alpha_k = \alpha_{k+1} \text{ for some } k,$$

and this is what we need to show.

Proof. Let τ be the transposition which interchanges k and $k+1$. Then $\alpha_{\sigma(i)} = \alpha_{\sigma(\tau(i))}$, and $\operatorname{sign}(\sigma) = -\operatorname{sign}(\sigma \circ \tau)$. Therefore,

$$\begin{aligned} & \sum_{\sigma \in \mathcal{S}_p} \operatorname{sign}(\sigma) \cdot \alpha_{\sigma(1)}(v_1) \cdot \alpha_{\sigma(2)}(v_2) \cdots \alpha_{\sigma(p)}(v_p) \\ = & -\sum_{\sigma \in \mathcal{S}_p} \operatorname{sign}(\sigma \circ \tau) \cdot \alpha_{\sigma\circ\tau(1)}(v_1) \cdot \alpha_{\sigma\circ\tau(2)}(v_2) \cdots \alpha_{\sigma\circ\tau(p)}(v_p). \end{aligned}$$

Note that

$$\left\{\sigma \circ \tau \mid \sigma \in \mathcal{S}_p\right\} = \left\{\sigma' \mid \sigma' \in \mathcal{S}_p\right\}.$$

This suggests performing the substitution $\sigma' := \sigma \circ \tau$, and our sum becomes

$$\begin{aligned} & \sum_{\sigma \in \mathcal{S}_p} \operatorname{sign}(\sigma) \cdot \alpha_{\sigma(1)}(v_1) \cdot \alpha_{\sigma(2)}(v_2) \cdots \alpha_{\sigma(p)}(v_p) \\ = & -\sum_{\sigma' \in \mathcal{S}_p} \operatorname{sign}(\sigma') \cdot \alpha_{\sigma'(1)}(v_1) \cdot \alpha_{\sigma'(2)}(v_2) \cdots \alpha_{\sigma'(p)}(v_p). \end{aligned}$$

These sums are clearly equal, and therefore equal to zero. □

It remains to show that our definition of $\omega(v_1, \ldots, v_p)$ yields an alternating map. For this, we must show that if $v_i = v_j$ for some $i \neq j$, then

$$\sum_{\sigma \in \mathcal{S}_p} \operatorname{sign}(\sigma) \cdot \alpha_{\sigma(1)}(v_1) \cdot \alpha_{\sigma(2)}(v_2) \cdots \alpha_{\sigma(p)}(v_p) = 0.$$

The proof is essentially identical, using the transposition which swaps i and j.

Up to some repetitive verifications that every alternating map arises this way, we have established the correspondence

$$\Lambda^p V^* \leftrightarrow \mathrm{Alt}^p(V).$$

Determinants

Based on our previous computations, it is natural to conjecture that

$$\varepsilon_1 \wedge \cdots \wedge \varepsilon_n(v_1, \ldots, v_n) \overset{?}{=} \det \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}.$$

This is easy to verify using the special case $p = n$ of our correspondence $\Lambda^n V^* \leftrightarrow \mathrm{Alt}^n(V)$. Note that $\det \in \mathrm{Alt}^n(V)$ since det is multilinear in the columns, and vanishes if any column is repeated. Also, $\mathrm{Alt}^n(V)$ is one-dimensional since

$$\dim \mathrm{Alt}^n(V) = \dim \Lambda^n V^* = \binom{n}{n} = 1.$$

Note that $\det \neq 0$ since $\det(e_1, \ldots, e_n) = 1$. Since $\mathrm{Alt}^n(V)$ is spanned by any nonzero element,

$$\varepsilon_1 \wedge \cdots \wedge \varepsilon_n(v_1, \ldots, v_n) = \lambda \det(v_1, \ldots, v_n)$$

for some constant λ. Plugging in $v_i = e_i$, we compute that $\lambda = 1$, and we have established our conjecture.

Properties of the exterior derivative

Recall that the space of degree p differential forms is defined as

$$\Omega^p(U) := C^\infty(U, \Lambda^p V^*).$$

We make the standard identification

$$\varepsilon_i = dx_i.$$

Since $\Lambda^p V^* = 0$ for $p > n$, we have $\Omega^p(U) = 0$ for $p > n$. Also, since $\Lambda^0 V^* = \mathbb{R}$, we have $\Omega^0(U) = C^\infty(U, \mathbb{R})$.

The de Rham complex is

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(U) \longrightarrow 0,$$

where the exterior derivative d is induced from the total derivative D. The exterior derivative may seem complicated at first, but in practice, it is much easier to compute with than D since it obeys several nice properties:

1. If $f \in \Omega^0(U)$, then

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

2. $d \circ d = 0$.

3. $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2$ if $\omega_1 \in \Omega^p(U)$.

4. Suppose $U_1 \subset \mathbb{R}^n$ and $U_2 \subset \mathbb{R}^m$ are open sets and $\phi : U_1 \to U_2$ is a smooth map. Then ϕ functorally induces maps $\phi^* : \Omega^p(U_2) \to \Omega^p(U_1)$ determined by

$$\phi^*(\omega \wedge \tau) = \phi^*(\omega) \wedge \phi^*(\tau),$$

5. $\phi^*(f) = f \circ \phi$ if $f \in \Omega^0(U_2)$,

6. $d\phi^*(\omega) = \phi^*(d\omega)$.

This gives us a recipe to compute ϕ^* as follows. Suppose that $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ are coordinate functions so that $x_i \in \Omega^0(U_1)$ and $y_j \in \Omega^0(U_2)$. Then the map ϕ may be viewed as a sequence of relations

$$\begin{aligned} y_1 &= \phi_1(x_1, \ldots, x_n) \\ y_2 &= \phi_2(x_1, \ldots, x_n) \\ &\vdots \\ y_m &= \phi_m(x_1, \ldots, x_m). \end{aligned}$$

Specifically, given the coordinate function $y_j \in \Omega^0(U_2)$, the coordinate functions $\phi_j \in \Omega^0(U_1)$ are obtained by

$$\phi_j := \phi^*(y_j) = y_j \circ \phi.$$

Example. Suppose $\phi(x_1, x_2) = (x_1^2, x_1 x_2)$. Then ϕ is given by $y_1 = x_1^2$, $y_2 = x_1 x_2$. We can compute ϕ^* via substitution, i.e.

$$\phi^*(y_1^2\, dy_2) = (x_1^2)^2\, d(x_1 x_2) = x_1^4\,(x_2\, dx_1 + x_1\, dx_2) = x_1^4 x_2\, dx_1 + x_1^5\, dx_2.$$

The de Rham complex as a generalization of vector calculus

The de Rham complex generalizes our old complex

$$C^\infty(U, \mathbb{R}) \overset{\vec{\nabla}}{\to} C^\infty(U, \mathbb{R}^3) \overset{\vec{\text{rot}}}{\to} C^\infty(U, \mathbb{R}^3) \overset{\text{div}}{\to} C^\infty(U, \mathbb{R}) \to 0.$$

For example, consider $\vec{\phi} \in C^\infty(U, \mathbb{R}^3)$, where $\vec{\phi} = \phi_1 e_1 + \phi_2 e_2 + \phi_3 e_3$. Then

$$\vec{\text{rot}}(\vec{\phi}) = \left(\frac{\partial \phi_3}{\partial x_2} - \frac{\partial \phi_2}{\partial x_3} \right) e_1 + \left(\frac{\partial \phi_1}{\partial x_3} - \frac{\partial \phi_3}{\partial x_1} \right) e_2 + \left(\frac{\partial \phi_2}{\partial x_1} - \frac{\partial \phi_1}{\partial x_2} \right) e_3.$$

Alternatively, the computation in the de Rham complex involves fewer worries about conventions. Consider

$$\phi = \phi_1 dx_1 + \phi_2 dx_2 + \phi_3 dx_3 \in \Omega^1(U).$$

Then

$$\begin{aligned}
d\phi &= d(\phi_1) \wedge dx_1 + \phi_1 \cancel{d dx_1} + d(\phi_2) \wedge dx_2 + d(\phi_3) \wedge dx_3 \in \Omega^2(U) \\
&= \left(\frac{\partial \phi_1}{\partial x_1} \cancel{dx_1} + \frac{\partial \phi_1}{\partial x_2} dx_2 + \frac{\partial \phi_1}{\partial x_3} dx_3 \right) \wedge dx_1 + \\
&\quad + \left(\frac{\partial \phi_2}{\partial x_1} dx_1 + \frac{\partial \phi_2}{\partial x_2} \cancel{dx_2} + \frac{\partial \phi_2}{\partial x_3} dx_3 \right) \wedge dx_2 + \\
&\quad + \left(\frac{\partial \phi_3}{\partial x_1} dx_1 + \frac{\partial \phi_3}{\partial x_2} dx_2 + \frac{\partial \phi_3}{\partial x_3} \cancel{dx_3} \right) \wedge dx_3 \\
&= \frac{\partial \phi_1}{\partial x_2} dx_2 \wedge dx_1 + \frac{\partial \phi_1}{\partial x_3} dx_3 \wedge dx_1 + \frac{\partial \phi_2}{\partial x_1} dx_1 \wedge dx_2 + \\
&\quad + \frac{\partial \phi_2}{\partial x_3} dx_3 \wedge dx_2 + \frac{\partial \phi_3}{\partial x_1} dx_1 \wedge dx_3 + \frac{\partial \phi_3}{\partial x_2} dx_2 \wedge dx_3 \\
&= \left(\frac{\partial \phi_3}{\partial x_2} - \frac{\partial \phi_2}{\partial x_3} \right) dx_2 \wedge dx_3 + \left(\frac{\partial \phi_1}{\partial x_3} - \frac{\partial \phi_3}{\partial x_1} \right) dx_3 \wedge dx_1 + \left(\frac{\partial \phi_2}{\partial x_1} - \frac{\partial \phi_1}{\partial x_2} \right) dx_1 \wedge dx_2.
\end{aligned}$$

This suggests that $\vec{\text{rot}}$ agrees with $d : \Omega^1(U) \to \Omega^2(U)$ when we identify $\Lambda^2 V^*$ with $\mathbb{R}^3$ via

$$\begin{aligned}
dx_2 \wedge dx_3 &\leftrightarrow e_1, \\
dx_3 \wedge dx_1 &\leftrightarrow e_2, \\
dx_1 \wedge dx_2 &\leftrightarrow e_3.
\end{aligned}$$