Contents

Problem Set 1

LaTeX

Mathematical typesetting is a helpful tool for communicating mathematics. Certain ill-conceived "Equation Editor"s are cumbersome and produce results which are not fit for human consumption. Thankfully, there is a widespread markup language (similar to HTML) called LaTeX (pronounced luh-Tech), which has been in use for decades. While the syntax of raw LaTeX is delicate and time-consuming to master, there are several editors which make its use relatively painless. If you intend to continue in mathematics, I highly recommend learning this skill, and I am happy to help you.

Personally, I use a free open-source multi-platform editor called LyX, available at `http://lyx.org`. It looks like a traditional word processor, it and has templates for choosing and positioning your desired mathematical symbols.

Problem 1.1 (*Required for grad students, optional for undergrads*)**.** Typeset your answer to one (or more) of the subsequent questions.

Perspectives on manifolds: stereographic projection

Some good quick references for stereographic projection are:

http://www.youtube.com/watch?v=6JgGKViQzbc

http://en.wikipedia.org/wiki/Stereographic_projection

We consider the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. As in Tuesday's lecture, observer #1 uses coordinates (X_1, Y_1) given by stereographic projection from the north pole $N = (0, 0, 1) \in \mathbb{S}^2$. We denote the projection function by $h_1 : \mathbb{S}^2 \to \mathbb{R}^2$, given in Wikipedia by

$$h_1(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right) \overset{\text{(def)}}{=:} (X_1, Y_1).$$

Observer #2 uses coordinates given by stereographic projection from the south pole $S = (0, 0, -1)$. This projection differs only by a reflection in the z coordinate $z \mapsto -z$. Thus, observer #2 uses the coordinates

$$h_2(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z} \right) =: (X_2, Y_2).$$

The "transition function" h_{21} takes as input the point (X_1, Y_1) measured by observer #1, and outputs the corresponding point (X_2, Y_2) for observer #2, where applicable. More precisely, if

$$\begin{aligned} (x, y, z) &\in \mathbb{S}^2, \\ h_1(x, y, z) &= (X_1, Y_1), \text{ and} \\ h_2(x, y, z) &= (X_2, Y_2), \end{aligned}$$

then $h_{21}(X_1, Y_1) = (X_2, Y_2)$.

Problem 1.2.

- Compute $h_{21}(X_1, Y_1)$.
- What's the significance of the domain of h_{21}?
  Clarification: There is a point at which h_{21} is undefined. What is the corresponding point on $\mathbb{S}^2$? In terms of the geometry of the sphere and the stereographic projections, what is the geometric reason why this point is not in the domain?

Note: Don't bother with deriving or memorizing the formulas for stereographic projection. Just take them as given.

Perspectives on cohomology: resolutions

Recall from Thursday's lecture that given a vector subspace $U \subset V^0$, a resolution of U is an exact sequence of linear maps

$$V^0 \to V^1 \to V^2 \to \cdots \to V^k \to 0$$

such that

- the kernel/nullspace of the map $V^0 \to V^1$ is U, and
- the image/range of $V^i \to V^{i+1}$ is the kernel of $V^{i+1} \to V^{i+2}$.

We don't yet need this level of formality. Instead, we think of the first map as describing relations which define U. The second map describes relations among those relations. The third map describes relations among those relations of relations, and so on.

For your convenience, here is the example from Thursday's lecture.

Let $U \subset \mathbb{R}^3$ be the subspace parameterized by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2t \\ 3t \\ 5t \end{pmatrix}$. This line can be characterized as the solution to the (redundant) equations

$$\begin{aligned} 3x - 2y + 0z &= 0, \\ 0x + 5y - 3z &= 0, \\ 3x + 3y - 3z &= 0, \end{aligned}$$

in other words, U is the kernel of the linear map

$$\mathbb{R}^3 \xrightarrow{\begin{pmatrix} 3 & -2 & 0 \\ 0 & 5 & -3 \\ 3 & 3 & -3 \end{pmatrix}} \mathbb{R}^3 .$$

The redundancy in this description is that the first equation plus the second equation minus the third equation equals zero, which is described by the matrix $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$. Thus we augment the chain by

$$\mathbb{R}^3 \xrightarrow{\begin{pmatrix} 3 & -2 & 0 \\ 0 & 5 & -3 \\ 3 & 3 & -3 \end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}} \mathbb{R} .$$

Since there is no redundancy in the matrix $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$ (i.e. it has full rank), we append the vector space $\mathbb{R}^0 = \left\{\vec{0}\right\}$ which we denote by 0, and our resolution is complete:

$$\mathbb{R}^3 \xrightarrow{\begin{pmatrix} 3 & -2 & 0 \\ 0 & 5 & -3 \\ 3 & 3 & -3 \end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}} \mathbb{R} \longrightarrow 0 .$$

One consequence is that the composition of the first two maps is zero:

$$\begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 0 & 5 & -3 \\ 3 & 3 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}.$$

Problem 1.3.

- Following the previous example, consider the planar subspace $U \subset \mathbb{R}^3$ spanned by the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$. Fill in the missing numbers to obtain a resolution of U.

$$\mathbb{R}^3 \xrightarrow{\begin{pmatrix} -3 & 6 & \square \\ \square & \square & 1 \end{pmatrix}} \mathbb{R}^2 \xrightarrow{\begin{pmatrix} \square & 3 \end{pmatrix}} \mathbb{R} \longrightarrow 0\ .$$

- Verify that the composition of the first two maps is zero.

A fundamental lemma on C^∞ Taylor polynomials

We need to get a handle on multivariable functions. A little-known-fact is that by appealing to Taylor polynomials, one can bypass many of the technical complications we would otherwise encounter. By doing a slightly lengthy calculus exercise now, we can save ourselves a lot of work in the future. Here is the result I will need.

Lemma. *Suppose that $f \in C^\infty(\mathbb{R})$, i.e. all the derivatives of f exist (and are continuous). Fix some constant $a \in \mathbb{R}$. For every non-negative integer k, there exists some function $S_k(x)$ $\underline{\text{also in } C^\infty(\mathbb{R})}$ so that*

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n + S_k(x)(x-a)^{k+1}.$$

For example,

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + S_3(x)x^4$$

for some $S_3 \in C^\infty(\mathbb{R})$. Why is this interesting? Note that when $x \neq 0$ we can solve for

$$S_3(x) = \frac{e^x - (1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3)}{x^4} \text{ when } x \neq 0.$$

The remarkable fact contained in this lemma is how $S_3(x)$ and *all* its derivatives extend to the problematic point $x = 0$. This allows us to use Taylor polynomials while remaining within the wonderful world of C^∞ functions.

Let's get to the proof!

Problem 1.4. Suppose that $f \in C^\infty(\mathbb{R})$, i.e. all the derivatives of f exist (and are continuous). Fix some constant $a \in \mathbb{R}$. For every non-negative integer k, define the k-th Taylor remainder

$$R_k(x) := \int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k \, dt.$$

1. Use the Fundamental Theorem of Calculus to show that $f(x) = f(a) + R_0(x)$.

2. Use integration by parts to show that

$$R_k(x) = \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} + R_{k+1}(x).$$

3. Use induction, and the preceeding facts, to prove that

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_k(x).$$

4. Use the substitution $t = a + u(x-a)$ to show that

$$R_k(x) = S_k(x)(x-a)^{k+1},$$

where

$$S_k(x) = \frac{1}{k!}\int_0^1 f^{(k+1)}(a + u(x-a))\,(1-u)^k\,du.$$

Observation: Because $f^{(k+1)}$ is a continuous function, the integrand

$$f^{(k+1)}(a + u(x-a))\,(1-u)^k$$

depends continuously on u. The definite Riemann integral of any continuous function

$$\int_0^1 (\text{continuous function of } u)\ du$$

is well-defined and finite. Therefore, $S_k(x)$ is a well-defined function.

5. Show that $S_k(x)$ is in $C^\infty(\mathbb{R})$, i.e. show that the j-th derivative

$$\frac{d^j}{dx^j} S_k(x)$$

is a well-defined function for all j. You may take for granted "differentiation under the integral sign," i.e. that

$$\frac{d}{dx}\int_0^1 g(x,u)\,du = \int_0^1 \frac{\partial}{\partial x} g(x,u)\,du$$

whenever $\frac{\partial}{\partial x} g(x,u)$ exists and is continuous.
Hint: It's unnecessary to explicitly compute $\frac{d^j}{dx^j} S_k(x)$. Just understand why it exists.

Problem Set 2

Functors

Refer to the definitions and examples on p. 27. Although we're skipping ahead here, this page is self-contained.

Take the category $\mathcal{C}$ to be

The category of open sets in Euclidean spaces, where the morphisms are the smooth maps.

Take the category $\mathcal{V}$ to be

The category of [real] vector spaces, where the morphisms are the linear maps.

We will verify some basic facts about the functor $C^\infty : \mathcal{C} \to \mathcal{V}$. This functor is defined on objects $U \in \mathcal{C}$ by

$$C^\infty(U) := C^\infty(U; \mathbb{R}).$$

We also need to specify how the C^∞ functor is defined on morphisms. This functor is contravariant, so for $\phi : U_1 \to U_2$, the induced map is some linear map of vector spaces $\phi^* : C^\infty(U_2) \to C^\infty(U_1)$. Specifically, it is defined by

$$\phi^*(f) := f \circ \phi.$$

Problem 2.1.

- Verify that ϕ^* is indeed a linear map of real vector spaces.
- Verify that the identity map $\mathrm{Id}_{U_1} : U_1 \to U_1$ induces the identity map $C^\infty(U_1) \to C^\infty(U_1)$.
- Verify that the composition map $\psi \circ \phi$

  $$U_1 \xrightarrow{\phi} U_2 \xrightarrow{\psi} U_3, \qquad U_1 \xrightarrow{\psi\circ\phi} U_3$$

  induces the composition $\phi^* \circ \psi^*$

  $$C^\infty(U_1) \xleftarrow{\phi^*} C^\infty(U_2) \xleftarrow{\psi^*} C^\infty(U_3), \qquad C^\infty(U_1) \xleftarrow{\phi^*\circ\psi^*} C^\infty(U_3).$$

  In other words, prove that $(\psi \circ \phi)^* = \phi^* \circ \psi^*$. (This is the sense in which a functor is a homomorphism with respect to composition of functions.)

Definition. A smooth (smooth = C^∞) map $\phi : U_1 \to U_2$ is said to be a *diffeomorphism* if there exists a smooth two-sided inverse map $\phi^{-1} : U_2 \to U_1$. Specifically,

$$\phi^{-1} \circ \phi = \mathrm{Id}_{U_1}, \text{ and}$$
$$\phi \circ \phi^{-1} = \mathrm{Id}_{U_2}.$$

Problem 2.2. Prove that a diffeomorphism induces an isomorphism of vector spaces.

(Note how your resulting proof is a formal consequence of the aforementioned properties. You don't need any additional facts about smooth functions or vector spaces. These sorts of vacuous category-theoretic proofs are affectionately called "abstract nonsense.")

Quotient spaces

Suppose V is a finite-dimensional real vector space, and $W \subset V$ is a vector subspace. Recall the definition of the quotient space V/W. (Wikipedia)

There is a natural "quotient map" $q : V \to V/W$ given by $v \mapsto [v]$.

Problem 2.3.

- Compute both the kernel and image of q. Use this to compute $\dim(V/W)$.
- Using q, construct an exact sequence which resolves the subspace W.

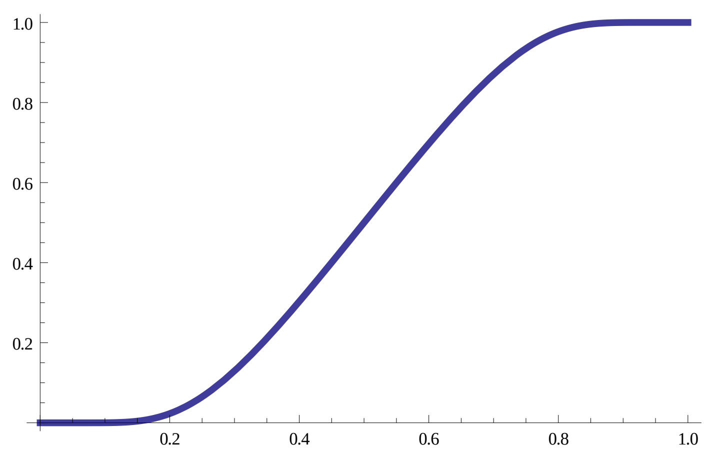
  Note: for deep technical reasons, such a resolution is not useful for defining cohomology.

Suppose further that V comes equipped with some Euclidean inner product denoted by $\langle \bullet, \bullet \rangle$. The orthocomplement of W is defined as

$$W^{\perp} := \{ v \in V \,|\, \langle v, w \rangle = 0 \text{ for all } w \in W \} .$$

Problem 2.4. Prove that the restriction of q to $W^{\perp}$ is an isomorphism.

Cutoff functions



It's useful to be able to craft functions which have carefully prescribed behavior. Specifically, we would like a "cutoff function" which interpolates between the constant functions zero and one. To an engineer, such a function looks like the voltage of a digital signal.

Specifically, we want a function $\psi(t)$ which satisfies

$$\begin{cases} \psi(t) = 0 & \text{if } t \leq 0, \\ \psi(t) = 1 & \text{if } t \geq 1, \\ 0 < \psi(t) < 1 & \text{otherwise.} \end{cases} \tag{1}$$

Constructing such a function can be reduced to a simpler problem as follows:

Problem 2.5. Show that if $\omega(t)$ satisfies

$$\begin{cases} \omega(t) = 0 & \text{if } t \le 0, \\ \omega(t) > 0 & \text{if } t > 0, \end{cases} \tag{2}$$

then the function

$$\psi(t) := \frac{\omega(t)}{\omega(t) + \omega(1-t)}$$

satisfies (1).

For example,

$$\omega(t) = t + |t| \implies \psi(t) = \begin{cases} 0 & \text{if } t \le 0, \\ t & \text{if } 0 \le t \le 1, \\ 1 & \text{if } t \ge 1. \end{cases}$$

This particular ω (and hence the corresponding ψ) is C^0, but not C^1. We would like to remain in the C^∞ realm, so we are led to the

Question Is it possible to have a C^∞ function satisfying (2) (and hence (1))?

Recall the following inclusions of function spaces:

$$\{\text{polynomials}\} \subset C^\infty(\mathbb{R}) \subset C^k(\mathbb{R}).$$

A more refined question is

Question How far left can we go in this diagram and still have a function satisfying (2)?

Recall that C^k denotes the space of functions whose k-th derivatives are continuous, where k is any nonnegative integer. It's easy to verify that

$$\omega_k(t) := (t + |t|)\, t^k \in C^k(\mathbb{R}) \tag{3}$$

satisfies (2). However, none of these functions are C^∞.

Problem 2.6. Prove that it's impossible for a polynomial function to satisfy (2). Are rational functions any different?

Now let's analyze the function

$$\omega_\infty(t) := \begin{cases} \exp(-1/t) & \text{if } t > 0, \\ 0 & \text{if } t \le 0. \end{cases}$$

Problem 2.7. Read and understand the proof of Lemma A.2 in Appendix A. (Hint: the limit comes from rewriting the n-th derivative $\omega_\infty^{(n)}(0)$ as a difference quotient.) Do not rewrite this proof. Instead, compute for ω_∞ the Maclaurin polynomials and remainder functions S_k, as described in Problem Set 1.

Decomposable tensors

Problem 2.8 (graduate level)**.** Prove that a tensor $t \in V \otimes W$ is decomposable iff the matrix of components has rank one. Hint: you should choose good basis vectors for both V and W.

Extra credit: More generally, show that the minimal number of terms required to express a tensor coincides with the matrix rank. Hint: choose your bases for V and W based on the singular value decomposition:

`http://en.wikipedia.org/wiki/Singular_value_decomposition#Geometric_meaning`

Tensor calculus

Let W be a vector space, which itself could be a tensor product. For $\phi \in C^\infty(U, W)$, we extend the definition of total derivative as follows. Choose a basis $\{w_j\}$ of W, and write $\phi = \sum \phi_j w_j$. Then

$$D\phi := \sum_{i,j} \frac{\partial \phi_j}{\partial x_i} \varepsilon_i \otimes w_j.$$

Problem 2.9. Compute

$$D\left((x_1^2 + x_2 x_3\, \varepsilon_1 \otimes \varepsilon_2) \otimes (x_2 + x_1 \varepsilon_1) \right).$$

Warning: the product rule for D doesn't work the way you might think, because the derivative always tensors in from the *left*, i.e. the definition of $D\phi$ involves $\varepsilon_i \otimes w_j$ rather than $w_j \otimes \varepsilon_i$.

Problem Set 3

Problem 3.1. Consider the permutation σ given by

n	1	2	3	4	5	6
$\sigma(n)$	4	3	1	5	6	2

Compute $\text{sign}(\sigma)$.

Problem 3.2. Compute

$$\varepsilon_4 \wedge \varepsilon_3 \wedge \varepsilon_1 \wedge \varepsilon_5 \wedge \varepsilon_6 \wedge \varepsilon_2 \left(e_1, e_2, e_3, e_4, e_5, e_6\right).$$

Problem 3.3. Compute

$$(\varepsilon_1 + 3\varepsilon_2 + 2\varepsilon_3) \wedge (9\varepsilon_1 + 5\varepsilon_2 + 7\varepsilon_3) \wedge (7\varepsilon_1 + 3\varepsilon_2 + 5\varepsilon_3).$$

Problem 3.4. Identify the components of

$$(v_1\varepsilon_1 + v_2\varepsilon_2 + v_3\varepsilon_3) \wedge (w_1\varepsilon_1 + w_2\varepsilon_2 + w_3\varepsilon_3)$$

with the components of the cross product $v \times w$.

Problem 3.5. For $\omega = \varepsilon_1 \wedge \varepsilon_2 + \varepsilon_3 \wedge \varepsilon_4 + \varepsilon_5 \wedge \varepsilon_6$, compute

- $\omega \wedge \omega$
- $\omega \wedge \omega \wedge \omega$
- $\omega \wedge \omega \wedge \omega \wedge \omega$

Problem (Graduate, M&T 2.10)**.** Let V be a 4-dimensional vector space and $\{\varepsilon_1, \dots, \varepsilon_4\}$ a basis for $\Lambda^1(V^*)$. Let $A = [a_{ij}]$ be a *skew-symmetric* matrix and define

$$\alpha = \sum_{i<j} a_{ij}\, \varepsilon_i \wedge \varepsilon_j.$$

Show that $\alpha \wedge \alpha = 0 \iff \det A = 0$. Say $\alpha \wedge \alpha = \lambda.\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4$. What is the relation between λ and $\det(A)$?

Hint:

$$\det \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = (af - be + cd)^2.$$

Problem 3.6. Consider the map $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\phi(x_1, x_2, x_3) = (x_1 x_2, x_2 x_3).$$

Suppose y_1, y_2 are coordinate functions on $\mathbb{R}^2$ with corresponding one-forms dy_1, dy_2. Let $\alpha \in \Omega^1(\mathbb{R}^2)$ be the one-form given by

$$\alpha = y_2 dy_1 - y_1^2 dy_2.$$

- Compute $\phi^*(\alpha) \in \Omega^1(\mathbb{R}^3)$.
- Compute $d\alpha$.
- Compute $\phi^*(d\alpha)$.
- Compute $d\phi^*(\alpha)$.

Problem 3.7 (M&T 3.1)**.** Show for an open set in $\mathbb{R}^2$ that the de Rham complex

$$\Omega^0(U) \to \Omega^1(U) \to \Omega^2(U) \to 0$$

is isomorphic to the complex

$$C^\infty(U, \mathbb{R}) \overset{\vec{\nabla}}{\to} C^\infty(U, \mathbb{R}^2) \overset{\text{rot}}{\to} C^\infty(U, \mathbb{R}) \to 0.$$

Analogously, show that for an open set in $\mathbb{R}^3$ the de Rham complex is isomorphic to

$$C^\infty(U, \mathbb{R}) \overset{\vec{\nabla}}{\to} C^\infty(U, \mathbb{R}^3) \overset{\vec{\text{rot}}}{\to} C^\infty(U, \mathbb{R}^3) \overset{\text{div}}{\to} C^\infty(U, \mathbb{R}) \to 0.$$

Problem Set 4

Let $\chi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be any operator obeying the following properties:

- $\mathbb{R}$-linearity: $\chi(\alpha f + \beta g) = \alpha\chi(f) + \beta\chi(g)$ for $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(\mathbb{R})$.
- Product rule: $\chi(f \cdot g) = \chi(f) \cdot g + f \cdot \chi(g)$.
- Identity: $\chi(x) = 1$, where $x \in C^\infty(\mathbb{R})$ denotes the identity function. (i.e. $\chi(f) = 1$ when $f(x) = x$.)

Problem 4.1.

- Prove that $\chi(1) = 0$, where $1 \in C^\infty(\mathbb{R})$ denotes the constant function with value 1.
- Prove that for any $a \in \mathbb{R}$,
  $$\chi(f)|_{x=a} = f'(a).$$
  Conclude that $\chi = \frac{d}{dx}$. Hint: Use a first-order Taylor polynomial.
- Suppose that the "Identity" property no longer holds. Rather than $\chi(x) = 1$, suppose $\chi(x) = g$ for some $g \in C^\infty(\mathbb{R})$. Generalize the above result.

Problem 4.2 (Poincaré lemma)**.** Consider

$$\omega = (x_2 - x_3)(dx_1 \wedge dx_2 + dx_3 \wedge dx_4) + (x_1 - x_4)(dx_1 \wedge dx_4 + dx_2 \wedge dx_3) \in \Omega^2(\mathbb{R}^4).$$

- Compute $d\omega$.
- Consider the map $\phi : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$, $\phi(\vec{x}, t) = t\vec{x}$. Compute that
  $$\begin{aligned}\phi^*(\omega) = t^3\omega + dt \wedge t^2\big((x_2 - x_3)(x_1\,dx_2 - x_2\,dx_1 + x_3\,dx_4 - x_4\,dx_3) + \\ + (x_1 - x_4)(x_1\,dx_4 - x_4\,dx_1 + x_2\,dx_3 - x_3\,dx_2)\big).\end{aligned}$$
- Following the proof of Theorem 3.15, compute $\eta := \hat{S}_1(\phi^*(\omega))$.
- Verify that $d\eta = \omega$.

Problem Set 5

First application of cohomology

Problem 5.1. Let's prove

Theorem. $\mathbb{R}^2 - \{0\}$ *is not diffeomorphic to any star-shaped region.*

First show that H^p is a functor.

Hint: Suppose $\phi : U_1 \to U_2$. Then verify the following statements. (The proofs should be about two lines each.) Look up the definitions of "cocycle," "coboundary" and "well-defined" in the "Background material" notes.

- If $\omega \in Z^p(U_2)$, then $\phi^*(\omega) \in Z^p(U_1)$.
- If $\omega \in B^p(U_2)$, then $\phi^*(\omega) \in B^p(U_1)$.
- If $[\omega] \in H^p(U_2)$, then $\phi^*([\omega]) := [\phi^*(\omega)]$ is well-defined.
- If ϕ is the identity on U, then ϕ^* is the identity on $H^p(U)$.
- $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

Now that we know H^p is a functor, explain why a diffeomorphism $\phi : U_1 \to U_2$ induces an isomorphism $\phi^* : H^p(U_2) \to H^p(U_1)$. (Again, two lines.)

Now consider

$$\alpha = \frac{x_1\, dx_2 - x_2\, dx_1}{x_1^2 + x_2^2} \in \Omega^1(\mathbb{R}^2 - \{0\}).$$

Verify that $\alpha \in Z^p(\mathbb{R}^2 - \{0\})$.

Define $\gamma : [0, 2\pi] \to \mathbb{R}^2 - \{0\}$ by $\gamma(t) = (\cos t, \sin t)$. Compute $\gamma^*(\alpha)$.

Based on the suggestive notation from your answer above, compute

$$\int \gamma^*(\alpha) \in \mathbb{R}.$$

Show that

$$\alpha \mapsto \int \gamma^*(\alpha)$$

is a linear map $\Omega^1(\mathbb{R}^2 - \{0\}) \to \mathbb{R}$.

Finally, use the Fundamental Theorem of Calculus to show that

$$[\alpha] \mapsto \int \gamma^*(\alpha)$$

is a well-defined map

$$H^1(\mathbb{R}^2 - \{0\}) \to \mathbb{R}.$$

Put together the above statements to conclude the theorem.

Maxwell's equations

Maxwell's equations describe the electromagnetic field. The electric field $\mathbf{E}(t) = \mathbf{E} = (E_1, E_2, E_3)$ and magnetic field $\mathbf{B}(t) = \mathbf{B} = (B_1, B_2, B_3)$ are vector fields on $\mathbb{R}^3$. Maxwell's equations are

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \cdot \mathbf{E} &= 4\pi\rho, \\ \nabla \times \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{J}, \end{aligned}$$

where ρ is the electric charge density and $\mathbf{J}$ is the electric current.

Spacetime is $\mathbb{R}^4$ with the coordinates (x_1, x_2, x_3, t). The electromagnetic field $F \in \Omega^2(\mathbb{R}^4)$ is

$$F := (E_1\, dx_1 + E_2\, dx_2 + E_3\, dx_3) \wedge dt + B_1\, dx_2 \wedge dx_3 + B_2\, dx_3 \wedge dx_1 + B_3\, dx_1 \wedge dx_2.$$

Problem 5.2. Show that

$$dF = 0 \iff \begin{cases} \nabla \cdot \mathbf{B} & = 0, \\ \nabla \times \mathbf{E} & = -\frac{\partial \mathbf{B}}{\partial t}. \end{cases}$$

Hint: Notice that the formula for F is invariant under the "cyclic permutation" $\begin{matrix} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 1 \end{matrix}$. Similarly, the component formulas for the dot and cross products have this same symmetry. You can greatly reduce your paper usage by exploiting cyclic permutations. For example,

$$\begin{aligned} F &= E_1\, dx_1 \wedge dt + B_1\, dx_2 \wedge dx_3 + \text{c.p.} \\ \nabla \times \mathbf{E} &= (\partial_2 E_3 - \partial_3 E_2)\vec{e}_1 + \text{c.p.} \end{aligned}$$

where c.p. denotes the sum over the remaining two cyclic permutations $\begin{matrix} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 1 \end{matrix}$, and $\begin{matrix} 1 \mapsto 3 \\ 2 \mapsto 1 \\ 3 \mapsto 2 \end{matrix}$.

To get the other two equations, we will need the Hodge star operator $\star$, which encodes the Minkowski structure of spacetime, and basically exchanges the electric and magnetic fields. Rather than define it, the result we need is:

$$\star F = (-B_1\, dx_1 - B_2\, dx_2 - B_3\, dx_3) \wedge dt + E_1\, dx_2 \wedge dx_3 + E_2\, dx_3 \wedge dx_1 + E_3\, dx_1 \wedge dx_2.$$

We define

$$J = \rho\, dx_1 \wedge dx_2 \wedge dx_3 - dt \wedge (J_1\, dx_2 \wedge dx_3 + J_2\, dx_3 \wedge dx_1 + J_3\, dx_1 \wedge dx_2).$$

Problem 5.3. Show that

$$d \star F = 4\pi J \iff \begin{cases} \nabla \cdot \mathbf{E} & = 4\pi\rho, \\ \nabla \times \mathbf{B} & = \frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{J}. \end{cases}$$

Therefore, Maxwell's equations are equivalent to:

$$dF = 0$$
$$d\star F = 4\pi J.$$

Problem 5.4. Use differential forms to prove that any solution of Maxwell's equations must satisfy the "conservation of charge" law $\dot{\rho} + \operatorname{div}(\mathbf{J}) = 0$.

Hint: compute dJ in two different ways. Your solution should be short.

The Poincaré lemma

Problem 5.5. Explain in two or fewer sentences (with equations) precisely why the electromagnetic field $F \in \Omega^2(\mathbb{R}^4)$ is determined by an "electromagnetic potential" $A \in \Omega^1(\mathbb{R}^4)$ satisfying $F = dA$.

Problem 5.6. Show that for any $A \in \Omega^1(\mathbb{R}^4)$ and $f \in \Omega^0(\mathbb{R}^4)$, the electromagnetic potentials A and $A + df$ represent the same electromagnetic field. Show that if $A_1, A_2 \in \Omega^1(\mathbb{R}^4)$ represent the same electromagnetic field, then $A_2 = A_1 + df$ for some $f \in \Omega^0(\mathbb{R}^4)$.

Remark. While the electromagnetic field is straightforward to measure, the "Aharonov-Bohm effect" shows that cohomology of the electromagnetic potential is also measurable. A solenoid running along the x_3-axis (and for all times t) generates an electromagnetic potential proportional to

$$A = \frac{x_1\,dx_2 - x_2\,dx_1}{x_1^2 + x_2^2} \in \Omega^1\left(\left\{(t, x_1, x_2, x_3) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 \neq 0\right\}\right).$$

Although the corresponding electromagnetic field is everywhere zero, the nonzero integral of Problem 5.1 leads to an observable quantum interference pattern.

Problem Set 6

The nerve of an open cover

As in Example 8.5 [M&T], Consider the two-sphere $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ with the atlas of six charts $\{(U_{\pm i}, h_{\pm i})\}_{i=1}^{3}$ where the $U_{\pm i}$ are given by

$$U_{+i} = \left\{x \in S^2 \mid x_i > 0\right\},$$
$$U_{-i} = \left\{x \in S^2 \mid x_i < 0\right\}.$$

Problem 6.1. Show that there is no quadruple intersection of the $U_{\pm i}$, i.e. show that the intersection of any four distinct $U_{\pm i}$ is empty.

The nerve of a cover is constructed as follows. For every open set U in our cover, we draw a vertex.

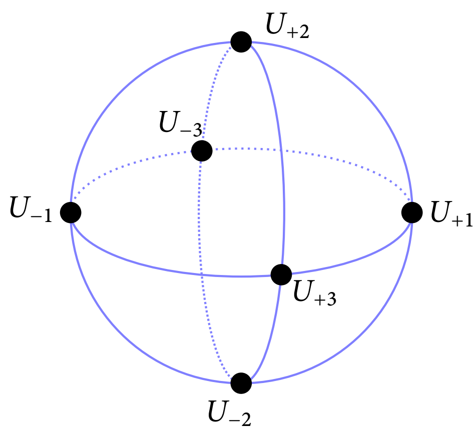
Draw a line segment for each pair of vertices for which the pairwise intersection is nonempty.

For each triple of vertices where the triple-intersection is nonempty, fill in the corresponding triangle.

For each quadruple of vertices where the quadruple-intersection is nonempty, fill in the corresponding tetrahedron.

(This construction continues to higher and higher dimensions.)

Problem 6.2. Sketch (some perspective of) the nerve of this cover $\{U_{\pm i}\}_{i=1}^{3}$, and briefly describe the shape. It will be helpful to think of each vertex as being located at the center of each open set. By Problem 7, we know that there are no tetrahedra to fill in.



Remark. A *good cover* for a manifold is an open cover for which all nonempty multiple-intersections are contractible. Such covers are good because the nerve encodes the topology of the manifold.

Definition of an atlas

Consider the set $X = \mathbb{R}$, and the standard atlas $\mathcal{A}_1 = \{(U_1, h_1)\} = \{(\mathbb{R}, \mathrm{id}_{\mathbb{R}})\}$.

Consider the function cube : $\mathbb{R} \to \mathbb{R}$ given by $\mathrm{cube}(x) = x^3$.

Problem 6.3. Is $\mathcal{A}_2 = \{(\mathbb{R}, \mathrm{cube})\}$ an atlas? Is it a smooth atlas? Why or why not?

Problem 6.4. Are $\mathcal{A}_1$ and $\mathcal{A}_2$ compatible? Why or why not?

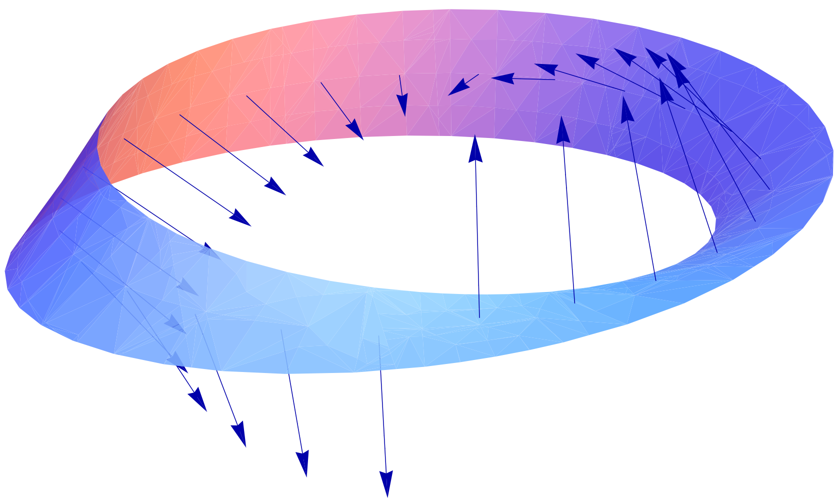
Recall that a diffeomorphism of manifolds $\phi : M_1 \to M_2$ is a smooth function with a smooth inverse.

Problem 6.5. Are the manifolds associated to $\mathcal{A}_1$ and $\mathcal{A}_2$ diffeomorphic? If so, give a diffeomorphism. If not, explain why.

Hint: In order to check that a map between manifolds is smooth, you must verify that the corresponding map under the coordinate charts is smooth. (See M&T 8.7 or Hitchin Chapter 1, 2.4.) I drew a corresponding diagram at some point in class.

Orientations

Definition. A smooth manifold M of dimension n is *orientable* if there exists some $\omega \in \Omega^n(M)$ such that ω is nowhere zero. Such an ω is called an *orientation form*. Two orientation forms ω_1 and ω_2 are *equivalent* if $\omega_1 = f \cdot \omega_2$ for some everywhere-positive $f \in \Omega^0(M)$. An *orientation* on M is an equivalence class of orientation forms. (M&T 9.8)



Suppose Σ^2 is a two-dimensional submanifold of $\mathbb{R}^3$, and $x \in \Sigma$. Let $\vec{n}$ be a nonzero vector which is normal to Σ at x. Define a function $T_x\Sigma \times T_x\Sigma \to \mathbb{R}$ denoted by $\vec{n}$ by

$$\vec{n}(\vec{t}_1, \vec{t}_2) := \vec{n} \cdot (\vec{t}_1 \times \vec{t}_2).$$

Problem 6.6. Show that $\vec{n}(\vec{t}_1, \vec{t}_2)$ is not the zero map on $T_x\Sigma$. (Hint: complete $\vec{n}$ to an orthonormal basis.) Show that this function is alternating, so that $\vec{n}$ defines a nonzero element of $\mathrm{Alt}^2(T_x\Sigma)$. (M&T 2.1)

Hint: $T_x\Sigma$ denotes the "tangent space" to Σ at the point x. Although we haven't discussed it yet in class, for these problems you should view it as the tangent plane at x. (It's a vector space, where all vectors have the initial point x.)

Suppose further that $\vec{n}$ extends to a nowhere vanishing field of smoothly-varying normal vectors along Σ.

Problem 6.7. According to the previous problem, what does such a vector field determine on Σ? (M&T 9.5) What does this say about the orientability of Σ?

Problem Set 7

As in the previous problem set, consider a surface $\Sigma^2 \subset \mathbb{R}^3$.

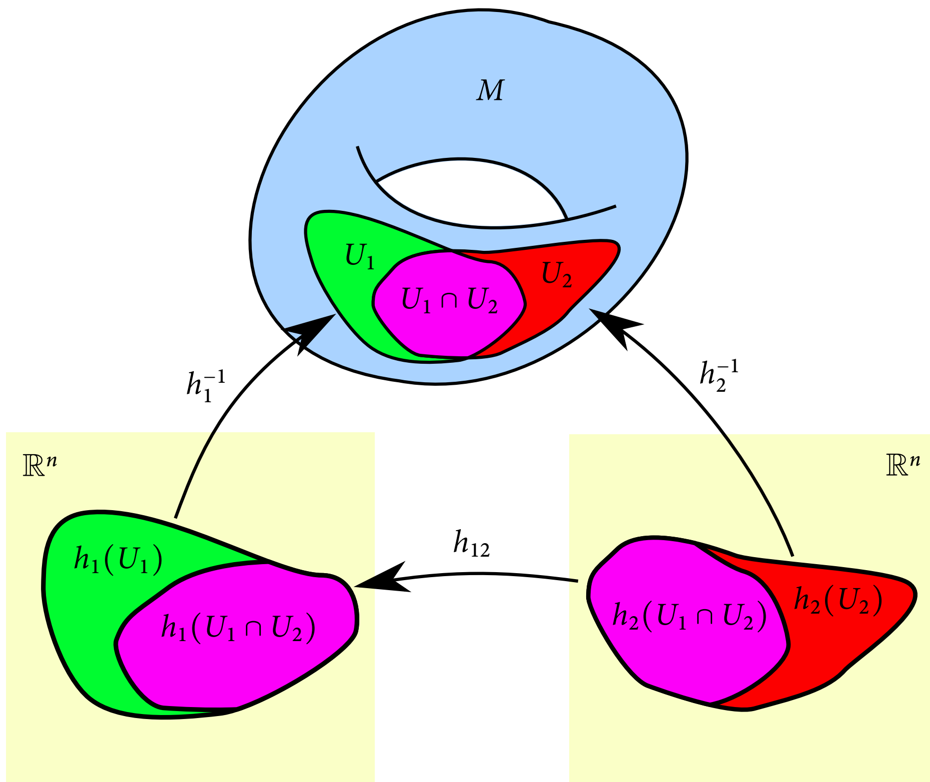
Problem 7.1. Given an orientation form ω on Σ, suggest (but don't bother to prove) an algorithm to produce a normal vector field along Σ. (Not much to do here, just read the hint and write out the procedure.)

Hints:

- Orthogonal projection $\pi : \mathbb{R}^3 \to T_x\Sigma$ induces a map $\pi^* : \Lambda^p(T_x^*\Sigma) \to \Lambda^p((\mathbb{R}^3)^*)$.
- In $\mathbb{R}^3$ we have a correspondence

$$e_1 \leftrightarrow \varepsilon_2 \wedge \varepsilon_3, \qquad e_2 \leftrightarrow \varepsilon_3 \wedge \varepsilon_1, \qquad e_3 \leftrightarrow \varepsilon_1 \wedge \varepsilon_2.$$

Now let's work out the transformation law for the orientation form.



$$\Big\Downarrow \Omega^n$$

$$\begin{array}{ccc} & \Omega^n(U_1 \cap U_2) & \\ \swarrow (h_1^{-1})^* & & (h_2^{-1})^* \searrow \\ \Omega^n(h_1(U_1 \cap U_2)) & \xrightarrow{(h_{12})^*} & \Omega^n(h_2(U_1 \cap U_2)) \end{array}$$

Suppose M^n is oriented, with orientation given by $\omega \in \Omega^n(M)$. Recall that $\dim \Lambda^n(V^*) = \binom{n}{n} = 1$, so in any local chart $h_1 : U_1 \to h_1(U_1) \subset \mathbb{R}^n$, the local representative $(h_1^{-1})^*(\omega)$ has only one component. This single component is the coefficient function of $dx_1 \wedge \cdots \wedge dx_n$, so

$$(h_1^{-1})^*(\omega) = f_1(x)\, dx_1 \wedge \cdots \wedge dx_n \in \Omega^n(h_1(U_1)),$$

for some $f \in \Omega^0(h_1(U_1))$.

Now let's consider the view from another chart $h_2 : U_2 \to h_2(U_2) \subset \mathbb{R}^n$. Here we have

$$(h_2^{-1})^*(\omega) = f_2(y)\, dy_1 \wedge \cdots \wedge dy_n \in \Omega^n(h_2(U_2)).$$

These two expressions are related via the transition function $h_{12} : h_2(U_1 \cap U_2) \to h_1(U_1 \cap U_2)$. Recall that h_{12} is a diffeomorphism. To simplify notation, we write $\phi := h_{12}$. The x-coordinates of $h_1(U_1 \cap U_2)$ are related to the y-coordinates of $h_2(U_1 \cap U_2)$ according to

$$x = \phi(y).$$

In components,

$$\begin{aligned} x_1 &= \phi_1(y_1, \ldots, y_n), \\ x_2 &= \phi_2(y_1, \ldots, y_n), \\ &\vdots \\ x_n &= \phi_n(y_1, \ldots, y_n). \end{aligned}$$

Correspondingly, we have the relation

$$f_2\, dy_1 \wedge \cdots \wedge dy_n = \phi^* \left(f_1\, dx_1 \wedge \cdots \wedge dx_n \right).$$

Problem 7.2. Compute the relationship between $\phi^*(f_1)$ and f_2 in terms of the partial derivatives $\frac{\partial \phi_i}{\partial y_j}$.

Hint: Use the identity

$$(a_{11}\varepsilon_1 + a_{12}\varepsilon_2 + \cdots + a_{1n}\varepsilon_n) \wedge (a_{21}\varepsilon_1 + \cdots + a_{2n}\varepsilon_n) \wedge \cdots \wedge (a_{n1}\varepsilon_1 + \cdots + a_{nn}\varepsilon_n) = \det(a_{ij})\varepsilon_1 \wedge \cdots \wedge \varepsilon_n.$$

Problem 7.3. Is it possible to have a point y where $f_1(\phi(y)) = 0$ but $f_2(y) \neq 0$? How about a point where $f_2(y) = 0$ but $f_1(\phi(y)) \neq 0$?

Hint: there is a corresponding transformation rule in the opposite direction given by

$$(\phi^{-1})^* \left(f_2\, dy_1 \wedge \cdots \wedge dy_n \right) = f_1\, dx_1 \wedge \cdots \wedge dx_n.$$

Let $\mathring{B}^n \subset M^n$ be a "small" open ball. There are two possible orientations on $\mathring{B}$, determined by the possible signs of f in a local coordinate expression for $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$. Given another ball $\mathring{B}'$ which (nicely) overlaps $\mathring{B}$, there is a unique orientation on $\mathring{B} \cup \mathring{B}'$ which is compatible with the orientation on $\mathring{B}$.

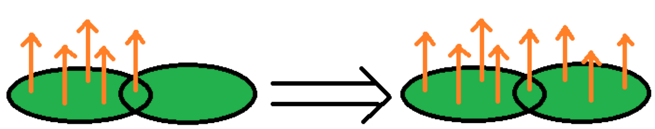


Figure 1: A unique compatible orientation for overlapping $\mathring{B}^2 \subset \mathbb{R}^3$

Problem 7.4. What is the property of $\mathring{B}$ which ensures that there are two possible orientations on $\mathring{B}$? Give an example of an open subset $U \subset \mathbb{R}^2$ which admits four possible orientations.

Problem 7.5. Argue that the Möbius strip is not orientable by using a sequence of balls which wind around the equator. Using a picture, explain how it's impossible to find a consistent orientation for the "last" ball which completes the loop.

Problem 7.6. Find a Möbius strip along the Klein bottle. Using the language of orientation forms, explain why this implies that the Klein bottle is non-orientable.

Supports

Recall that the support of a function f is defined to be

$$\text{supp}(f) := \text{closure}\left(\{x | f(x) \neq 0\}\right).$$

Problem 7.7. (Graduate students, or undergrads who like point-set topology) Prove that if $U \subset \mathbb{R}^n$ is open, $f, \psi \in C^\infty(U)$, and $\text{supp}(\psi) \subset U$, then $f\psi$ extends by zero to $f\psi \in C^\infty(\mathbb{R}^n)$.

Hint: to show $f\psi \in C^\infty(\mathbb{R}^n)$, it suffices to show that for every point $p \in \mathbb{R}^n$, there is an open ball $\mathring{B} \ni p$ such that $f\psi \in C^\infty(\mathring{B})$. Consider separately the two cases $p \in \text{supp}(\psi)$ and $p \notin \text{supp}(\psi)$.

Now consider the smooth function

$$\psi(x) := \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Define $f(x) := \frac{1}{\psi(x)^2}$.

Problem 7.8.

- What is the domain of f? Make a rough sketch of the graph of f.
- For any constant ε, what is the support of $\psi(x - \varepsilon)$?
- For which values of ε is $\text{supp}(\psi(x - \varepsilon)) \subset \text{domain}(f)$?
- In the above case, give a rough qualitative sketch of the graph of $\psi(x - \varepsilon)f(x)$.
  Hint: http://goo.gl/clWTz
- What happens when ε is borderline, so that $\text{supp}(\psi(x-\varepsilon)) \not\subset \text{domain}(f)$? Does $\psi(x-\varepsilon)f(x)$ extend to a smooth function over $\mathbb{R}$?
  Hint: to make your intuition rigorous, recall that if a function g is continuous, then by definition it satisfies $g(x_0) = \lim_{x \to x_0} g(x)$ for all x_0.

Tangent and cotangent vectors on S^2

Let $p \in S^2$ be the point $p = (\sqrt{\frac{1}{2}}, 0, \sqrt{\frac{1}{2}})$. Consider the curve $\gamma(t) = (\cos(t + \frac{\pi}{4}), 0, \sin(t + \frac{\pi}{4}))$, and $f \in C^\infty(S^2)$ given by $f(x, y, z) = z$.

Problem 7.9. Compute $(f \circ \gamma)'(0)$. Determine both $\gamma'(0)$ and $df|_p$ in both the (X_1, Y_1) and (X_2, Y_2) coordinate systems (see Pset 1, and the Wikipedia article for stereographic projection). In each coordinate system, verify that your answers are consistent with $df|_p \cdot \gamma'(0) = (f \circ \gamma)'(0)$.

Problem Set 8

Orientations and linear maps

The manifold $\mathbb{R}^n$ has two possible orientations, determined by

$$\pm dx_1 \wedge \cdots \wedge dx_n.$$

Let's wean ourselves away from using normal vectors to visualize orientations.

We begin with the well-known fact from linear algebra that every invertible square matrix can be written as a product of elementary matrices `http://en.wikipedia.org/wiki/Elementary_matrix`. Correspondingly, every invertible transformation of $\mathbb{R}^n$ is a composition of the following transformations:

- Coordinate swap $x_i \leftrightarrow x_j$,
- Scale transformation $x_i \to kx_i$ with $k \neq 0$,
- Skew transformation $x_i \to x_i + kx_j$ with $i \neq j$.

Note that a coordinate swap corresponds to reflection about the plane $x_i = x_j$. Also, a scale transformation with $k = -1$ corresponds to a reflection about the $x_i = 0$ plane. After possibly composing with this $k = -1$ reflection, we can assume that $k > 0$. Thus we consider four types of transformations:

- Coordinate swap $x_i \leftrightarrow x_j$,
- Scale transformation $x_i \to -x_i$,
- Scale transformation $x_i \to kx_i$ with $k > 0$,
- Skew transformation $x_i \to x_i + kx_j$ with $i \neq j$.

Problem 8.1. How do each of these four elementary transformations affect the two orientations of $\mathbb{R}^n$? How do the first two "reflective" transformations compare with the last two "non-reflective" transformations?

In your head, think about the properties you would expect from a sensible notion of "orientation." According to what you would expect, do the above results make sense?

Cotangent vectors

Consider $p \in M^n$ and $\alpha \in T_p^* M$. Suppose that in a local coordinate chart,

$$\alpha = \alpha_1 \, dx_1|_p + \cdots + \alpha_n \, dx_n|_p$$

for some coefficients where $\alpha_i \in \mathbb{R}$.

Problem 8.2. Construct some $f \in \Omega^0(M)$ so that $df|_p = \alpha$.

Note: make sure f is defined on all of M, and not just within the coordinate chart.

Problem Set 9

Derivations

Recall that in local coordinates, a vector field V looks like

$$V = V_1 \frac{\partial}{\partial x_1} + \cdots + V_n \frac{\partial}{\partial x_n},$$

for smooth functions V_i. This notation suggests that V acts as a directional derivative operator. Specifically, given a function $f \in \Omega^0(M)$, we define $Vf \in \Omega^0(M)$, which is given locally by

$$Vf = V_1 \frac{\partial f}{\partial x_1} + \cdots + V_n \frac{\partial f}{\partial x_n}.$$

Problem 9.1. Prove that V is a *derivation* on $\Omega^0(M)$, i.e.

$$V(\alpha f + \beta g) = \alpha V f + \beta V g \text{ for } \alpha, \beta \in \mathbb{R} \text{ and } f, g \in \Omega^0(M),$$
$$V(fg) = g\, Vf + f\, Vg \text{ for } f, g \in \Omega^0(M).$$

Hint: Since a function is determined by its values at each point, it suffices to check these formulas locally. There is no need to reverify the transformation rule. (In class I already verified in painstaking detail that the transformation rule checks out.)

Problem 9.2. Referring to your answer for Problem Set 4, Problem 1, show that in the case $M = \mathbb{R}$, every derivation determines a vector field on M. Thus for $M = \mathbb{R}$, "vector fields on $\mathbb{R}$" are equivalent to "derivations on $\Omega^0(\mathbb{R})$."

Note: it's not difficult to prove that for a general manifold M, "vector fields on M" are equivalent to "derivations on $\Omega^0(M)$." The proof is a slight modification of your answer for Problem Set 4, Problem 1.

Riemannian metric

In class I outlined how for $S^2 \subset \mathbb{R}^3$, the metric $dx^2 + dy^2 + dz^2$ pulls back under stereographic projection to an induced metric in the X_1, Y_1 coordinates, which computations show is

$$\left(\frac{2}{1 + X_1^2 + Y_1^2}\right)^2 \left(dX_1^2 + dY_1^2\right).$$

Consider the path $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (\underbrace{t}_{X_1}, \underbrace{0}_{Y_1})$.

Let's do some computations using this metric.

Problem 9.3. Based on the geometry of S^2, explain what the length of γ should be. Then use the induced metric to verify your answer.

Problem 9.4. Using the inner product associated with the metric, show that the vector fields $\frac{\partial}{\partial X_1}$ and $\frac{\partial}{\partial Y_1}$ are orthogonal at every point.

Problem 9.5. For any point $(X_1, Y_1) \in \mathbb{R}^2$, use the metric to compute the side lengths for the infinitesimal rectangle with base vector $(\Delta X_1)\frac{\partial}{\partial X_1}$ and height vector $(\Delta Y_1)\frac{\partial}{\partial Y_1}$, for $\Delta X_1, \Delta Y_1 \in \mathbb{R}$. Then compute the area of this infinitesimal rectangle. Compute the area of S^2 by summing the areas of these infinitesimal rectangles while letting $\Delta X_1 \to dX_1$ and $\Delta Y_1 \to dY_1$. Finally, compare the integral you evaluated with the general volume formula

$$\text{vol} = \int \sqrt{\det(g_{ij})}\, dx_1 \cdots dx_n,$$

where the g_{ij} are the components of the metric.