Derivations

Recall that in local coordinates, a vector field V looks like

$$V = V_1 \frac{\partial}{\partial x_1} + \cdots + V_n \frac{\partial}{\partial x_n},$$

for smooth functions V_i. This notation suggests that V acts as a directional derivative operator. Specifically, given a function $f \in \Omega^0(M)$, we define $Vf \in \Omega^0(M)$, which is given locally by

$$Vf = V_1 \frac{\partial f}{\partial x_1} + \cdots + V_n \frac{\partial f}{\partial x_n}.$$

Problem 1. Prove that V is a *derivation* on $\Omega^0(M)$, i.e.

$$V(\alpha f + \beta g) = \alpha V f + \beta V g \text{ for } \alpha, \beta \in \mathbb{R} \text{ and } f, g \in \Omega^0(M),$$
$$V(fg) = g\, Vf + f\, Vg \text{ for } f, g \in \Omega^0(M).$$

Hint: Since a function is determined by its values at each point, it suffices to check these formulas locally. There is no need to reverify the transformation rule. (In class I already verified in painstaking detail that the transformation rule checks out.)

Problem 2. Referring to your answer for Problem Set 4, Problem 1, show that in the case $M = \mathbb{R}$, every derivation determines a vector field on M. Thus for $M = \mathbb{R}$, "vector fields on $\mathbb{R}$" are equivalent to "derivations on $\Omega^0(\mathbb{R})$."

Note: it's not difficult to prove that for a general manifold M, "vector fields on M" are equivalent to "derivations on $\Omega^0(M)$." The proof is a slight modification of your answer for Problem Set 4, Problem 1.

Riemannian metric

In class I outlined how for $S^2 \subset \mathbb{R}^3$, the metric $dx^2 + dy^2 + dz^2$ pulls back under stereographic projection to an induced metric in the X_1, Y_1 coordinates, which computations show is

$$\left(\frac{2}{1 + X_1^2 + Y_1^2} \right)^2 \left(dX_1^2 + dY_1^2 \right).$$

Consider the path $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (\underbrace{t}_{X_1}, \underbrace{0}_{Y_1})$.

Let's do some computations using this metric.

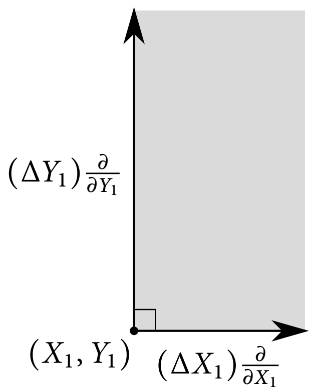
Problem 3. Based on the geometry of S^2, explain what the length of γ should be. Then use the induced metric to verify your answer.

Problem 4. Using the inner product associated with the metric, show that the vector fields $\frac{\partial}{\partial X_1}$ and $\frac{\partial}{\partial Y_1}$ are orthogonal at every point.

Problem 5. For any point $(X_1, Y_1) \in \mathbb{R}^2$, use the metric to compute the side lengths for the infinitesimal rectangle with base vector $(\Delta X_1)\frac{\partial}{\partial X_1}$ and height vector $(\Delta Y_1)\frac{\partial}{\partial Y_1}$, for $\Delta X_1, \Delta Y_1 \in \mathbb{R}$. Then compute the area of this infinitesimal rectangle. Compute the area of S^2 by summing the areas of these infinitesimal rectangles while letting $\Delta X_1 \to dX_1$ and $\Delta Y_1 \to dY_1$. Finally, compare the integral you evaluated with the general volume formula

$$\text{vol} = \int \sqrt{\det(g_{ij})}\, dx_1 \cdots dx_n,$$

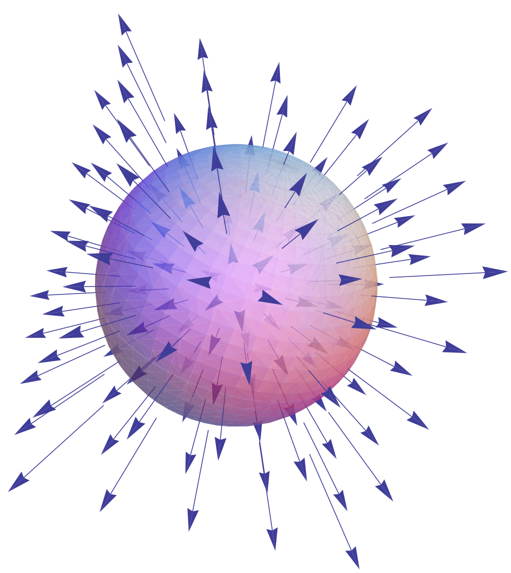
where the g_{ij} are the components of the metric.



Tuesday option

Consider the vector field on $\mathbb{R}^3$ given by

$$V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$



Let $e : S^2 \hookrightarrow \mathbb{R}^3$ denote the standard embedding, and $h_i : S^2 \to \mathbb{R}^2$ for $i \in \{1, 2\}$ be the standard coordinate charts with coordinates (X_i, Y_i).

$$\mathbb{R}^2 \xrightarrow{h_i^{-1}} S^2 \xhookrightarrow{e} \mathbb{R}^3 .$$

Problem 6. Show that V corresponds to a differential form ω on $\mathbb{R}^3$ such that $e^*(\omega)$ is an orientation form on S^2.

Hint: On the individual tangent spaces, the map e^* is a left-inverse to the map π^* from Problem Set 7, Problem 1. The remainder of that problem should suggest what to choose for ω. Now to show that $e^*(\omega)$ is an orientation form on S^2, it suffices to show that $e^*(\omega)$ is nonzero in each of our coordinate charts h_1 and h_2. For the lazy:

$$h_i^{-1}(X_i, Y_i) = \left(\underbrace{\frac{2X_i}{1 + X_i^2 + Y_i^2}}_{x}, \underbrace{\frac{2Y_i}{1 + X_i^2 + Y_i^2}}_{y}, \underbrace{(-1)^{i-1}\frac{-1 + X_i^2 + Y_i^2}{1 + X_i^2 + Y_i^2}}_{z} \right),$$

$$\begin{aligned}
(e \circ h_i^{-1})^*(dx) &= d\frac{2X_i}{1 + X_i^2 + Y_i^2} \\
&= \frac{2}{(1 + X_i^2 + Y_i^2)^2}\left((1 - X_i^2 + Y_i^2)\, dX_i - 2X_iY_i\, dY_i\right), \\
(e \circ h_i^{-1})^*(dy) &= \frac{2}{(1 + X_i^2 + Y_i^2)^2}\left(-2X_iY_i\, dX_i + (1 + X_i^2 - Y_i^2)\, dY_i\right), \\
(e \circ h_i^{-1})^*(dz) &= (-1)^{i-1}\frac{4}{(1 + X_i^2 + Y_i^2)^2}\,(X_i\, dX_i + Y_i\, dY_i),
\end{aligned}$$

$$\begin{aligned}
\left(-2X_iY_i\, dX_i + (1 + X_i^2 - Y_i^2)\, dY_i\right) \wedge (X_i\, dX_i + Y_i\, dY_i) &= -X_i(1 + X_i^2 + Y_i^2)\, dX_i \wedge dY_i, \\
(X_i\, dX_i + Y_i\, dY_i) \wedge \left((1 - X_i^2 + Y_i^2)\, dX_i - 2X_iY_i\, dY_i\right) &= -Y_i\left(1 + X_i^2 + Y_i^2\right)\, dX_i \wedge dY_i, \\
\left((1 - X_i^2 + Y_i^2)\, dX_i - 2X_iY_i\, dY_i\right) \wedge & \\
\wedge \left(-2X_iY_i\, dX_i + (1 + X_i^2 - Y_i^2)\, dY_i\right) &= -(-1 + X_i^2 + Y_i^2)\left(1 + X_i^2 + Y_i^2\right)\, dX_i \wedge dY_i,
\end{aligned}$$

$$(-1)^{i-1}\left(-16X_i^2 - 16Y_i^2 - 4\left(-1 + X_i^2 + Y_i^2\right)^2\right) = 4(-1)^i(1 + X_i^2 + Y_i^2)^2.$$

(Now you don't need to use a computer!)

Problem 7. Tweak our standard atlas of S^2 to make it positive with respect to $e^*(\omega)$, and compute $\int_{(S^2,[e^*(\omega)])} e^*(\omega)$. Tweak the atlas to make it positive with respect to $-e^*(\omega)$, and compute $\int_{(S^2,[-e^*(\omega)])} e^*(\omega)$.

Hint: You don't need a partition of unity to compute these integrals. Since a single coordinate chart covers S^2 minus a point, and because the missing point has measure zero, it suffices to integrate $e^*(\omega)$ unmodified in a single coordinate chart.

Problem 8. Compute $d\omega$ and $\int_{B^3} d\omega$. Comparing the result to your answer for 7, explain the coincidence in terms of Stokes' Theorem $\int_{\partial M} e^*(\omega) = \int_M d\omega$. (Don't worry about signs and orientation conventions.)