As in the previous problem set, consider a surface $\Sigma^2 \subset \mathbb{R}^3$.

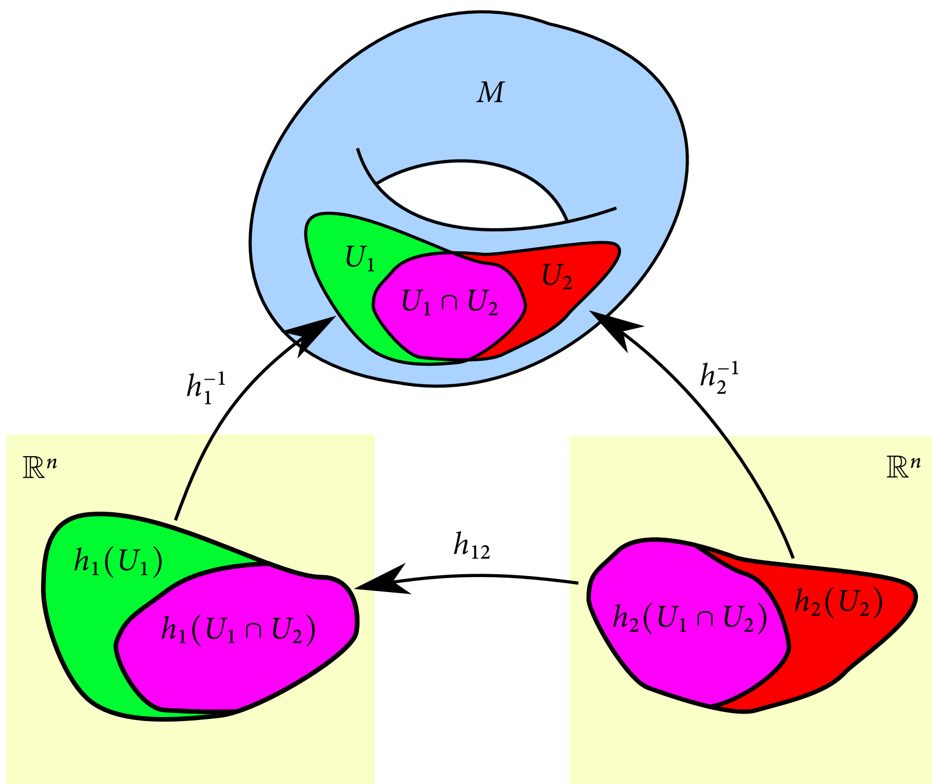
Problem 1. Given an orientation form ω on Σ, suggest (but don't bother to prove) an algorithm to produce a normal vector field along Σ. (Not much to do here, just read the hint and write out the procedure.)

Hints:

- Orthogonal projection $\pi : \mathbb{R}^3 \to T_x\Sigma$ induces a map $\pi^* : \Lambda^p(T_x^*\Sigma) \to \Lambda^p((\mathbb{R}^3)^*)$.
- In $\mathbb{R}^3$ we have a correspondence

$$e_1 \leftrightarrow \varepsilon_2 \wedge \varepsilon_3, \qquad e_2 \leftrightarrow \varepsilon_3 \wedge \varepsilon_1, \qquad e_3 \leftrightarrow \varepsilon_1 \wedge \varepsilon_2.$$

Now let's work out the transformation law for the orientation form.



$$\Big\Downarrow \Omega^n$$

$$\begin{array}{ccc} & \Omega^n(U_1 \cap U_2) & \\ \swarrow (h_1^{-1})^* & & \searrow (h_2^{-1})^* \\ \Omega^n(h_1(U_1 \cap U_2)) & \xrightarrow{(h_{12})^*} & \Omega^n(h_2(U_1 \cap U_2)) \end{array}$$

Suppose M^n is oriented, with orientation given by $\omega \in \Omega^n(M)$. Recall that $\dim \Lambda^n(V^*) = \binom{n}{n} = 1$, so in any local chart $h_1 : U_1 \to h_1(U_1) \subset \mathbb{R}^n$, the local representative $(h_1^{-1})^*(\omega)$ has only one component. This single component is the coefficient function of $dx_1 \wedge \cdots \wedge dx_n$, so

$$(h_1^{-1})^*(\omega) = f_1(x)\, dx_1 \wedge \cdots \wedge dx_n \in \Omega^n(h_1(U_1)),$$

for some $f \in \Omega^0(h_1(U_1))$.

Now let's consider the view from another chart $h_2 : U_2 \to h_2(U_2) \subset \mathbb{R}^n$. Here we have

$$(h_2^{-1})^*(\omega) = f_2(y)\, dy_1 \wedge \cdots \wedge dy_n \in \Omega^n(h_2(U_2)).$$

These two expressions are related via the transition function $h_{12} : h_2(U_1 \cap U_2) \to h_1(U_1 \cap U_2)$. Recall that h_{12} is a diffeomorphism. To simplify notation, we write $\phi := h_{12}$. The x-coordinates of $h_1(U_1 \cap U_2)$ are related to the y-coordinates of $h_2(U_1 \cap U_2)$ according to

$$x = \phi(y).$$

In components,

$$\begin{aligned} x_1 &= \phi_1(y_1, \ldots, y_n), \\ x_2 &= \phi_2(y_1, \ldots, y_n), \\ &\vdots \\ x_n &= \phi_n(y_1, \ldots, y_n). \end{aligned}$$

Correspondingly, we have the relation

$$f_2\, dy_1 \wedge \cdots \wedge dy_n = \phi^* \left(f_1\, dx_1 \wedge \cdots \wedge dx_n \right).$$

Problem 2. Compute the relationship between $\phi^*(f_1)$ and f_2 in terms of the partial derivatives $\frac{\partial \phi_i}{\partial y_j}$.

Hint: Use the identity

$$(a_{11}\varepsilon_1 + a_{12}\varepsilon_2 + \cdots + a_{1n}\varepsilon_n) \wedge (a_{21}\varepsilon_1 + \cdots + a_{2n}\varepsilon_n) \wedge \cdots \wedge (a_{n1}\varepsilon_1 + \cdots + a_{nn}\varepsilon_n) = \det(a_{ij})\varepsilon_1 \wedge \cdots \wedge \varepsilon_n.$$

Problem 3. Is it possible to have a point y where $f_1(\phi(y)) = 0$ but $f_2(y) \neq 0$? How about a point where $f_2(y) = 0$ but $f_1(\phi(y)) \neq 0$?

Hint: there is a corresponding transformation rule in the opposite direction given by

$$(\phi^{-1})^* \left(f_2\, dy_1 \wedge \cdots \wedge dy_n \right) = f_1\, dx_1 \wedge \cdots \wedge dx_n.$$

Let $\mathring{B}^n \subset M^n$ be a "small" open ball. There are two possible orientations on $\mathring{B}$, determined by the possible signs of f in a local coordinate expression for $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$. Given another ball $\mathring{B}'$ which (nicely) overlaps $\mathring{B}$, there is a unique orientation on $\mathring{B} \cup \mathring{B}'$ which is compatible with the orientation on $\mathring{B}$.

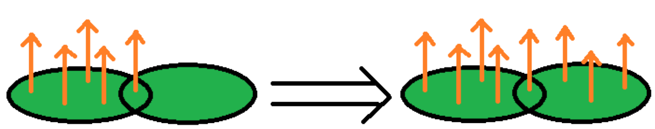


Figure 1: A unique compatible orientation for overlapping $\mathring{B}^2 \subset \mathbb{R}^3$

Problem 4. What is the property of $\mathring{B}$ which ensures that there are two possible orientations on $\mathring{B}$? Give an example of an open subset $U \subset \mathbb{R}^2$ which admits four possible orientations.

Problem 5. Argue that the Möbius strip is not orientable by using a sequence of balls which wind around the equator. Using a picture, explain how it's impossible to find a consistent orientation for the "last" ball which completes the loop.

Problem 6. Find a Möbius strip along the Klein bottle. Using the language of orientation forms, explain why this implies that the Klein bottle is non-orientable.

Supports

Recall that the support of a function f is defined to be

$$\operatorname{supp}(f) := \operatorname{closure}\left(\{x | f(x) \neq 0\}\right).$$

Problem 7. (Graduate students, or undergrads who like point-set topology) Prove that if $U \subset \mathbb{R}^n$ is open, $f, \psi \in C^\infty(U)$, and $\operatorname{supp}(\psi) \subset U$, then $f\psi$ extends by zero to $f\psi \in C^\infty(\mathbb{R}^n)$.

Hint: to show $f\psi \in C^\infty(\mathbb{R}^n)$, it suffices to show that for every point $p \in \mathbb{R}^n$, there is an open ball $\mathring{B} \ni p$ such that $f\psi \in C^\infty(\mathring{B})$. Consider separately the two cases $p \in \operatorname{supp}(\psi)$ and $p \notin \operatorname{supp}(\psi)$.

Now consider the smooth function

$$\psi(x) := \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Define $f(x) := \frac{1}{\psi(x)^2}$.

Problem 8.

- What is the domain of f? Make a rough sketch of the graph of f.
- For any constant ε, what is the support of $\psi(x - \varepsilon)$?
- For which values of ε is $\operatorname{supp}(\psi(x - \varepsilon)) \subset \operatorname{domain}(f)$?
- In the above case, give a rough qualitative sketch of the graph of $\psi(x - \varepsilon) f(x)$.
  Hint: http://goo.gl/clWTz
- What happens when ε is borderline, so that $\operatorname{supp}(\psi(x-\varepsilon)) \not\subset \operatorname{domain}(f)$? Does $\psi(x-\varepsilon)f(x)$ extend to a smooth function over $\mathbb{R}$?
  Hint: to make your intuition rigorous, recall that if a function g is continuous, then by definition it satisfies $g(x_0) = \lim_{x \to x_0} g(x)$ for all x_0.

Tangent and cotangent vectors on S^2

Let $p \in S^2$ be the point $p = (\sqrt{\frac{1}{2}}, 0, \sqrt{\frac{1}{2}})$. Consider the curve $\gamma(t) = (\cos(t + \frac{\pi}{4}), 0, \sin(t + \frac{\pi}{4}))$, and $f \in C^\infty(S^2)$ given by $f(x, y, z) = z$.

Problem 9. Compute $(f \circ \gamma)'(0)$. Determine both $\gamma'(0)$ and $df|_p$ in both the (X_1, Y_1) and (X_2, Y_2) coordinate systems (see Pset 1, and the Wikipedia article for stereographic projection). In each coordinate system, verify that your answers are consistent with $df|_p \cdot \gamma'(0) = (f \circ \gamma)'(0)$.