Let $\chi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be any operator obeying the following properties:

- $\mathbb{R}$-linearity: $\chi(\alpha f + \beta g) = \alpha\chi(f) + \beta\chi(g)$ for $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(\mathbb{R})$.
- Product rule: $\chi(f \cdot g) = \chi(f) \cdot g + f \cdot \chi(g)$.
- Identity: $\chi(x) = 1$, where $x \in C^\infty(\mathbb{R})$ denotes the identity function. (i.e. $\chi(f) = 1$ when $f(x) = x$.)

Problem 1.

- Prove that $\chi(1) = 0$, where $1 \in C^\infty(\mathbb{R})$ denotes the constant function with value 1.
- Prove that for any $a \in \mathbb{R}$,
$$\chi(f)|_{x=a} = f'(a).$$
Conclude that $\chi = \frac{d}{dx}$. Hint: Use a first-order Taylor polynomial.
- Suppose that the "Identity" property no longer holds. Rather than $\chi(x) = 1$, suppose $\chi(x) = g$ for some $g \in C^\infty(\mathbb{R})$. Generalize the above result.

Problem 2 (Poincaré lemma)**.** Consider

$$\omega = (x_2 - x_3)(dx_1 \wedge dx_2 + dx_3 \wedge dx_4) + (x_1 - x_4)(dx_1 \wedge dx_4 + dx_2 \wedge dx_3) \in \Omega^2(\mathbb{R}^4).$$

- Compute $d\omega$.
- Consider the map $\phi : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$, $\phi(\vec{x}, t) = t\vec{x}$. Compute that
$$\begin{aligned}\phi^*(\omega) = t^3\omega + dt \wedge t^2\big((x_2 - x_3)(x_1\, dx_2 - x_2\, dx_1 + x_3\, dx_4 - x_4\, dx_3) + \\ + (x_1 - x_4)(x_1\, dx_4 - x_4\, dx_1 + x_2\, dx_3 - x_3\, dx_2)\big).\end{aligned}$$
- Following the proof of Theorem 3.15, compute $\eta := \hat{S}_1(\phi^*(\omega))$.
- Verify that $d\eta = \omega$.

Tuesday option

Problem 3. Let's prove

Theorem. $\mathbb{R}^2 - \{0\}$ *is not diffeomorphic to any star-shaped region.*

First show that H^p is a functor.

Hint: Suppose $\phi : U_1 \to U_2$. Then verify the following statements. (The proofs should be about two lines each.) Look up the definitions of "cocycle," "coboundary" and "well-defined" in the "Background material" notes.

- If $\omega \in Z^p(U_2)$, then $\phi^*(\omega) \in Z^p(U_1)$.
- If $\omega \in B^p(U_2)$, then $\phi^*(\omega) \in B^p(U_1)$.
- If $[\omega] \in H^p(U_2)$, then $\phi^*([\omega]) := [\phi^*(\omega)]$ is well-defined.
- If ϕ is the identity on U, then ϕ^* is the identity on $H^p(U)$.
- $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

Now that we know H^p is a functor, explain why a diffeomorphism $\phi : U_1 \to U_2$ induces an isomorphism $\phi^* : H^p(U_2) \to H^p(U_1)$. (Again, two lines.)

Now consider

$$\alpha = \frac{x_1\, dx_2 - x_2\, dx_1}{x_1^2 + x_2^2} \in \Omega^1(\mathbb{R}^2 - \{0\}).$$

Verify that $\alpha \in Z^p(\mathbb{R}^2 - \{0\})$.

Define $\gamma : [0, 2\pi] \to \mathbb{R}^2 - \{0\}$ by $\gamma(t) = (\cos t, \sin t)$. Compute $\gamma^*(\alpha)$.

Based on the suggestive notation from your answer above, compute

$$\int \gamma^*(\alpha) \in \mathbb{R}.$$

Show that

$$\alpha \mapsto \int \gamma^*(\alpha)$$

is a linear map $\Omega^1(\mathbb{R}^2 - \{0\}) \to \mathbb{R}$.

Finally, use the Fundamental Theorem of Calculus to show that

$$[\alpha] \mapsto \int \gamma^*(\alpha)$$

is a well-defined map

$$H^1(\mathbb{R}^2 - \{0\}) \to \mathbb{R}.$$

Put together the above statements to conclude the theorem.