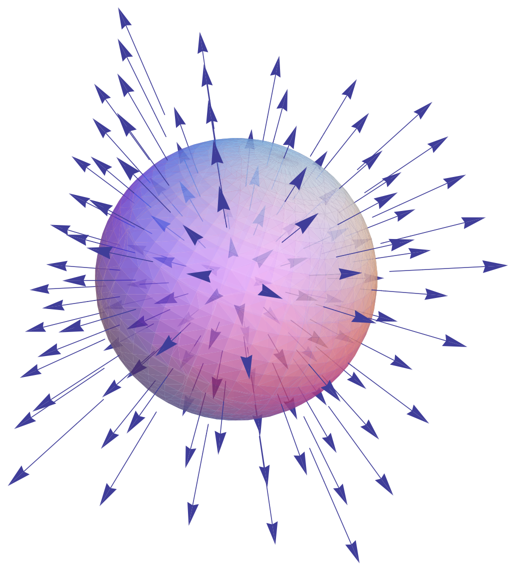
Consider the vector field on $\mathbb{R}^3$ given by

$$V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$



Let $e : S^2 \hookrightarrow \mathbb{R}^3$ denote the standard embedding, and $h_i : S^2 \to \mathbb{R}^2$ for $i \in \{1, 2\}$ be the standard coordinate charts with coordinates (X_i, Y_i).

$$\mathbb{R}^2 \xrightarrow{h_i^{-1}} S^2 \overset{e}{\hookrightarrow} \mathbb{R}^3 .$$

Problem 1. Show that V corresponds to a differential form ω on $\mathbb{R}^3$ such that $e^*(\omega)$ is an orientation form on S^2.

Hint: On the individual tangent spaces, the map e^* is a left-inverse to the map π^* from Problem Set 7, Problem 1. The remainder of that problem should suggest what to choose for ω. Now to show that $e^*(\omega)$ is an orientation form on S^2, it suffices to show that $e^*(\omega)$ is nonzero in each of our coordinate charts h_1 and h_2. For the lazy:

$$h_i^{-1}(X_i, Y_i) = \left(\underbrace{\frac{2X_i}{1 + X_i^2 + Y_i^2}}_{x}, \underbrace{\frac{2Y_i}{1 + X_i^2 + Y_i^2}}_{y}, \underbrace{(-1)^{i-1}\frac{-1 + X_i^2 + Y_i^2}{1 + X_i^2 + Y_i^2}}_{z} \right),$$

$$
\begin{aligned}
(e \circ h_i^{-1})^*(dx) &= d\frac{2X_i}{1 + X_i^2 + Y_i^2} \\
&= \frac{2}{(1 + X_i^2 + Y_i^2)^2}\left((1 - X_i^2 + Y_i^2)\, dX_i - 2X_i Y_i\, dY_i\right), \\
(e \circ h_i^{-1})^*(dy) &= \frac{2}{(1 + X_i^2 + Y_i^2)^2}\left(-2X_i Y_i\, dX_i + (1 + X_i^2 - Y_i^2)\, dY_i\right), \\
(e \circ h_i^{-1})^*(dz) &= (-1)^{i-1}\frac{4}{(1 + X_i^2 + Y_i^2)^2}\,(X_i\, dX_i + Y_i\, dY_i),
\end{aligned}
$$

$$
\begin{aligned}
\left(-2X_i Y_i\, dX_i + (1 + X_i^2 - Y_i^2)\, dY_i\right) \wedge (X_i\, dX_i + Y_i\, dY_i) &= -X_i(1 + X_i^2 + Y_i^2)\, dX_i \wedge dY_i, \\
(X_i\, dX_i + Y_i\, dY_i) \wedge \left((1 - X_i^2 + Y_i^2)\, dX_i - 2X_i Y_i\, dY_i\right) &= -Y_i\left(1 + X_i^2 + Y_i^2\right) dX_i \wedge dY_i, \\
\left((1 - X_i^2 + Y_i^2)\, dX_i - 2X_i Y_i\, dY_i\right) \wedge & \\
\wedge \left(-2X_i Y_i\, dX_i + (1 + X_i^2 - Y_i^2)\, dY_i\right) &= -(-1 + X_i^2 + Y_i^2)\left(1 + X_i^2 + Y_i^2\right) dX_i \wedge dY_i,
\end{aligned}
$$

$$
(-1)^{i-1}\left(-16X_i^2 - 16Y_i^2 - 4\left(-1 + X_i^2 + Y_i^2\right)^2\right) = 4(-1)^i(1 + X_i^2 + Y_i^2)^2.
$$

(Now you don't need to use a computer!)

Problem 2. Tweak our standard atlas of S^2 to make it positive with respect to $e^*(\omega)$, and compute $\int_{(S^2,[e^*(\omega)])} e^*(\omega)$. Tweak the atlas to make it positive with respect to $-e^*(\omega)$, and compute $\int_{(S^2,[-e^*(\omega)])} e^*(\omega)$.

Hint: You don't need a partition of unity to compute these integrals. Since a single coordinate chart covers S^2 minus a point, and because the missing point has measure zero, it suffices to integrate $e^*(\omega)$ unmodified in a single coordinate chart.

Problem 3. Compute $d\omega$ and $\int_{B^3} d\omega$. Comparing the result to your answer for 2, explain the coincidence in terms of Stokes' Theorem $\int_{\partial M} e^*(\omega) = \int_M d\omega$. (Don't worry about signs and orientation conventions.)

Stokes' revenge

The remainder of this problem set should be viewed as an extension of Problem 1 from Problem Set 5.

Consider $[\omega] \in H^p(M^n)$. Let K^p be any closed (compact w/o boundary) oriented manifold. Suppose we also have a fixed smooth map $k : K \to M$. We define the pairing

$$
\langle k, [\omega] \rangle := \int_K k^*(\omega) \in \mathbb{R}.
$$

Problem 4. Show that the pairing $\langle k, [\omega]\rangle$ is well-defined. Show that pairing with k gives a linear map $H^p(M) \to \mathbb{R}$.

Problem 5. Show that there is a value of $p \in \mathbb{R}$ for which

$$\omega_p := \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2 + y^2 + z^2)^p} \in \Omega^2(\mathbb{R}^3 - \{0\})$$

is closed.

Hint:

$$\frac{\partial}{\partial x} \frac{x}{(x^2 + y^2 + z^2)^p} = \frac{1 - 2p\frac{x^2}{x^2+y^2+z^2}}{(x^2 + y^2 + z^2)^p}.$$

Problem 6. Up to sign, what is $\pm \int_{S^2} \omega_p$?

Hint: If $e : S^2 \hookrightarrow \mathbb{R}^3$ is the standard embedding, then $e^*(\omega_p)$ is independent of p. (Why?) Now recall your computation from Problem Set 9, Problem 7.

Problem 7. Using the results of Problems 4, 5, and 2, conclude something about $H^2(\mathbb{R}^3 - \{0\})$.

Tuesday option

Suppose $k_0, k_1 : K \to M$ are smoothly homotopic, through some smooth homotopy $\tilde{k} : [0, 1] \times K \to M$.

Problem 8. Show that $\langle k_0, [\omega]\rangle = \langle k_1, [\omega]\rangle$.

Hint: $\partial([0, 1] \times K) = K \sqcup \overline{K}$, where $\overline{K}$ denotes the manifold K equipped with the opposite orientation.

More generally, suppose K_0 and K_1 are closed oriented p-dimensional manifolds. We say that two maps $k_0 : K_0 \to M$ and $k_1 : K_1 \to M$ are *bordant* when there is some $(p + 1)$-dimensional manifold $\tilde{K}$ (which is compact, oriented, and *with* boundary) and a map $\tilde{k} : \tilde{K} \to M$ such that $\partial\tilde{K} = K_0 \sqcup \overline{K}_1$ and $\tilde{k}|_{K_i} = k_i$ for $i = 1, 2$.

Problem 9. Show that $\langle k_0, [\omega]\rangle = \langle k_1, [\omega]\rangle$ whenever k_0 and k_1 are bordant. Conclude that $\langle [k], [\omega]\rangle$ is well-defined, where $[k]$ denotes the bordism class of k.

Note: The notion of *homology* $H_p(M)$ is very similar to that of bordism. Roughly speaking, one considers finite linear combinations of bordisms, where the K's are required to be simplices (hypertriangles). The "universal coefficient theorem" states that $H^p(M) = H_p(M)^*$, so that cohomology is the dual vector space of homology.