Calculus on Manifolds
Problem Set 1 (with notes)
Due Thursday, Jan. 12, 2012, beginning of class

Introduction

Since we are not yet in the textbook, I wanted to provide some reference material to help you along. The high page count is primarily due to the notes I have included, plus the way I broke down the final problem into several sub-parts. Feel free to let me know if you find any typos, if you get stuck, or if you would like clarification on anything.

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LaTeX

Mathematical typesetting is a helpful tool for communicating mathematics. Certain ill-conceived "Equation Editor"s are cumbersome and produce results which are not fit for human consumption. Thankfully, there is a widespread markup language (similar to HTML) called LaTeX (pronounced luh-Tech), which has been in use for decades. While the syntax of raw LaTeX is delicate and time-consuming to master, there are several editors which make its use relatively painless. If you intend to continue in mathematics, I highly recommend learning this skill, and I am happy to help you.

Personally, I use a free open-source multi-platform editor called LyX, available at http://lyx.org. It looks like a traditional word processor, it and has templates for choosing and positioning your desired mathematical symbols.

Problem 1 (*Required for grad students, optional for undergrads*). Typeset your answer to one (or more) of the subsequent questions.

Perspectives on manifolds: stereographic projection

Some good quick references for stereographic projection are:

http://www.youtube.com/watch?v=6JgGKViQzbc

http://en.wikipedia.org/wiki/Stereographic_projection

We consider the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. As in Tuesday's lecture, observer #1 uses coordinates (X_1, Y_1) given by stereographic projection from the north pole $N = (0,0,1) \in \mathbb{S}^2$. We denote the projection function by $h_1 : \mathbb{S}^2 \to \mathbb{R}^2$, given in Wikipedia by

$$h_1(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \overset{(\text{def})}{=:} (X_1, Y_1).$$

Observer #2 uses coordinates given by stereographic projection from the south pole $S = (0, 0, -1)$. This projection differs only by a reflection in the z coordinate $z \mapsto -z$. Thus, observer #2 uses the coordinates

$$h_2(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z} \right) =: (X_2, Y_2).$$

The "transition function" h_{21} takes as input the point (X_1, Y_1) measured by observer #1, and outputs the corresponding point (X_2, Y_2) for observer #2, where applicable. More precisely, if

$$\begin{aligned} (x, y, z) &\in \mathbb{S}^2, \\ h_1(x, y, z) &= (X_1, Y_1), \text{ and} \\ h_2(x, y, z) &= (X_2, Y_2), \end{aligned}$$

then $h_{21}(X_1, Y_1) = (X_2, Y_2)$.

Problem 2.

- Compute $h_{21}(X_1, Y_1)$.
- What's the significance of the domain of h_{21}?
  Clarification: There is a point at which h_{21} is undefined. What is the corresponding point on $\mathbb{S}^2$? In terms of the geometry of the sphere and the stereographic projections, what is the geometric reason why this point is not in the domain?

Note: Don't bother with deriving or memorizing the formulas for stereographic projection. Just take them as given.

Perspectives on cohomology: resolutions

Recall from Thursday's lecture that given a vector subspace $U \subset V^0$, a resolution of U is an exact sequence of linear maps

$$V^0 \to V^1 \to V^2 \to \cdots \to V^k \to 0$$

such that

- the kernel/nullspace of the map $V^0 \to V^1$ is U, and
- the image/range of $V^i \to V^{i+1}$ is the kernel of $V^{i+1} \to V^{i+2}$.

We don't yet need this level of formality. Instead, we think of the first map as describing relations which define U. The second map describes relations among those relations. The third map describes relations among those relations of relations, and so on.

For your convenience, here is the example from Thursday's lecture.

Let $U \subset \mathbb{R}^3$ be the subspace parameterized by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2t \\ 3t \\ 5t \end{pmatrix}$. This line can be characterized as the solution to the (redundant) equations

$$\begin{aligned} 3x - 2y + 0z &= 0, \\ 0x + 5y - 3z &= 0, \\ 3x + 3y - 3z &= 0, \end{aligned}$$

in other words, U is the kernel of the linear map

$$\mathbb{R}^3 \xrightarrow{\begin{pmatrix} 3 & -2 & 0 \\ 0 & 5 & -3 \\ 3 & 3 & -3 \end{pmatrix}} \mathbb{R}^3 .$$

The redundancy in this description is that the first equation plus the second equation minus the third equation equals zero, which is described by the matrix $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$. Thus we augment the chain by

$$\mathbb{R}^3 \xrightarrow{\begin{pmatrix} 3 & -2 & 0 \\ 0 & 5 & -3 \\ 3 & 3 & -3 \end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}} \mathbb{R} .$$

Since there is no redundancy in the matrix $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$ (i.e. it has full rank), we append the vector space $\mathbb{R}^0 = \{\vec{0}\}$ which we denote by 0, and our resolution is complete:

$$\mathbb{R}^3 \xrightarrow{\begin{pmatrix} 3 & -2 & 0 \\ 0 & 5 & -3 \\ 3 & 3 & -3 \end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}} \mathbb{R} \longrightarrow 0 .$$

One consequence is that the composition of the first two maps is zero:

$$\begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 0 & 5 & -3 \\ 3 & 3 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}.$$

Problem 3.

- Following the previous example, consider the planar subspace $U \subset \mathbb{R}^3$ spanned by the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$. Fill in the missing numbers to obtain a resolution of U.

$$\mathbb{R}^3 \xrightarrow{\begin{pmatrix} -3 & 6 & \square \\ \square & \square & 1 \end{pmatrix}} \mathbb{R}^2 \xrightarrow{\begin{pmatrix} \square & 3 \end{pmatrix}} \mathbb{R} \longrightarrow 0 .$$

- Verify that the composition of the first two maps is zero.

A fundamental lemma on C^∞ Taylor polynomials

We need to get a handle on multivariable functions. A little-known-fact is that by appealing to Taylor polynomials, one can bypass many of the technical complications we would otherwise encounter. By doing a slightly lengthy calculus exercise now, we can save ourselves a lot of work in the future. Here is the result I will need.

Lemma. *Suppose that $f \in C^\infty(\mathbb{R})$, i.e. all the derivatives of f exist (and are continuous). Fix some constant $a \in \mathbb{R}$. For every non-negative integer k, there exists some function $S_k(x)$ also in $C^\infty(\mathbb{R})$ so that*

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n + S_k(x)(x-a)^{k+1}.$$

For example,

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + S_3(x)x^4$$

for some $S_3 \in C^\infty(\mathbb{R})$. Why is this interesting? Note that when $x \neq 0$ we can solve for

$$S_3(x) = \frac{e^x - \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3\right)}{x^4} \text{ when } x \neq 0.$$

The remarkable fact contained in this lemma is how $S_3(x)$ and *all* its derivatives extend to the problematic point $x = 0$. This allows us to use Taylor polynomials while remaining within the wonderful world of C^∞ functions.

Let's get to the proof!

Problem 4. Suppose that $f \in C^\infty(\mathbb{R})$, i.e. all the derivatives of f exist (and are continuous). Fix some constant $a \in \mathbb{R}$. For every non-negative integer k, define the k-th Taylor remainder

$$R_k(x) := \int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k \, dt.$$

1. Use the Fundamental Theorem of Calculus to show that $f(x) = f(a) + R_0(x)$.

2. Use integration by parts to show that

$$R_k(x) = \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} + R_{k+1}(x).$$

3. Use induction, and the preceeding facts, to prove that

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_k(x).$$

4. Use the substitution $t = a + u(x-a)$ to show that

$$R_k(x) = S_k(x)(x-a)^{k+1},$$

where

$$S_k(x) = \frac{1}{k!}\int_0^1 f^{(k+1)}(a + u(x-a))\,(1-u)^k \, du.$$

Observation: Because $f^{(k+1)}$ is a continuous function, the integrand

$$f^{(k+1)}(a + u(x - a))\,(1 - u)^k$$

depends continuously on u. The definite Riemann integral of any continuous function

$$\int_0^1 (\text{continuous function of } u)\; du$$

is well-defined and finite. Therefore, $S_k(x)$ is a well-defined function.

5. Show that $S_k(x)$ is in $C^\infty(\mathbb{R})$, i.e. show that the j-th derivative

$$\frac{d^j}{dx^j} S_k(x)$$

is a well-defined function for all j. You may take for granted "differentiation under the integral sign," i.e. that

$$\frac{d}{dx} \int_0^1 g(x,u)\, du = \int_0^1 \frac{\partial}{\partial x} g(x,u)\, du$$

whenever $\frac{\partial}{\partial x} g(x,u)$ exists and is continuous.
Hint: It's unnecessary to explicitly compute $\frac{d^j}{dx^j} S_k(x)$. Just understand why it exists.