- Cohomology of S^n
- Proof of Mayer-Vietoris
- Other cohomology theories
- Course evals

Last time we saw that each row is a short exact sequence (SES):

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & \Omega^0(M) & \longrightarrow & \Omega^0(U)\oplus\Omega^0(V) & \longrightarrow & \Omega^0(U\cap V) & \longrightarrow & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(U)\oplus\Omega^1(V) & \longrightarrow & \Omega^1(U\cap V) & \longrightarrow & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & \Omega^n(M) & \longrightarrow & \Omega^n(U)\oplus\Omega^n(V) & \longrightarrow & \Omega^n(U\cap V) & \longrightarrow & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{1}
$$

The columns are chain complexes ($d^2 = 0$). The diagram commutes, because d commutes with restrictions. This diagram is abbreviated by

$$0 \longrightarrow \Omega^\bullet(M) \longrightarrow \Omega^\bullet(U)\oplus\Omega^\bullet(V) \longrightarrow \Omega^\bullet(U\cap V) \longrightarrow 0\,, \tag{2}$$

which is a SES of chain complexes.

I claim that associated to any SES of chain complexes, there is an associated long exact sequence (LES)

$$
\begin{aligned}
0 &\longrightarrow H^0(M) \longrightarrow H^0(U) \oplus H^0(V) \longrightarrow H^0(U \cap V) \\
&\longrightarrow H^1(M) \longrightarrow H^1(U) \oplus H^1(V) \longrightarrow H^1(U \cap V) \\
&\longrightarrow \cdots \\
&\cdots \\
&\longrightarrow H^n(M) \longrightarrow H^n(U) \oplus H^n(V) \longrightarrow H^n(U \cap V) \\
&\longrightarrow 0
\end{aligned}
\tag{3}
$$

We defer two major questions:

- What is the connecting morphism $H^p(U \cap V) \to H^{p+1}(M)$?
- Why is this exact?

Counting dimensions in an exact sequence

Claim 1. In an exact sequence, the dimension of any space is the sum of the ranks in and out.

$$\xrightarrow[r_{\text{in}}]{} \mathbb{R}^{r_{\text{in}}+r_{\text{out}}} \xrightarrow[r_{\text{out}}]{} \tag{4}$$

Proof.

$$V^{i-1} \xrightarrow{L^{i-1}} V^i \xrightarrow{L^i} V^{i+1} \tag{5}$$

By the rank-nullity theorem,

$$
\begin{aligned}
\dim V^i &= \dim\ker(L^i) + \dim\operatorname{im}(L^i) \\
&= \dim\operatorname{im}(L^{i-1}) + \dim\operatorname{im}(L^i) \\
&= \operatorname{rk}(L^{i-1}) + \operatorname{rk}(L^{i-1}).
\end{aligned}
\tag{6}
$$

□

Example. Suppose $M = U \sqcup V$ is a disjoint union, so $U \cap V = \varnothing$. Then $H^p(U \cap V) = H^p(\varnothing) = 0$, so we get

$$
\begin{aligned}
0 &\longrightarrow H^0(M) \longrightarrow H^0(U) \oplus H^0(V) \longrightarrow 0 \\
&\longrightarrow H^1(M) \longrightarrow H^1(U) \oplus H^1(V) \longrightarrow 0 \\
&\longrightarrow \cdots \\
&\qquad\qquad\qquad\qquad \cdots \\
&\longrightarrow H^n(M) \longrightarrow H^n(U) \oplus H^n(V) \longrightarrow 0 \\
&\longrightarrow 0
\end{aligned}
\tag{7}
$$

which consists of several levels of

$$0 \xrightarrow[0]{} H^p(M) \xrightarrow[r]{L^p} H^p(U) \oplus H^p(V) \xrightarrow[0]{} 0 \tag{8}$$

with indicated ranks. (Any map to/from zero must have rank zero.)

$$
\begin{aligned}
\dim H^p(M) = 0 + r &\implies L^p \text{ injective}, \\
H^p(U) \oplus H^p(V) = r + 0 &\implies L^p \text{ surjective},
\end{aligned}
\tag{9}
$$

thus L^p is an isomorphism, so if $M = U \sqcup V$ then

$$H^p(M) \cong H^p(U) \oplus H^p(V). \tag{10}$$

Cohomology of S^n

$S^0 = \{\pm 1\}$, so

$$H^p(S^0) = H^p(\mathrm{pt}) \oplus H^p(\mathrm{pt}) = \begin{cases} \mathbb{R}^2 & \text{if } p = 0, \\ 0 & \text{else.} \end{cases} \tag{11}$$

Henceforth assume $n \geq 1$. We will proceed by induction on n, to show

$$H^p(S^n) = \begin{cases} \mathbb{R} & \text{if } p = 0, n \\ 0 & \text{else.} \end{cases} \tag{12}$$

Let $N, S = (0, \ldots, 0, \pm 1)$. Take $U = S^n - \{N\}$, $V = S^n - \{S\}$, so that $U \cup V = S^n$. Observe that U, V are contractible, and that $U \cap V = S^n - \{N, S\}$ deformation-retracts to the equatorial S^{n-1}. Thus $H^p(U) \cong H^p(V) \cong H^p(\mathrm{pt})$, and $H^p(U \cap V) = H^p(S^{n-1})$.

We may assume that $H^p(S^n) = \mathbb{R}^{b_p(S^n)}$ for some (possibly infinite) Betti numbers $b_p(S^n)$. Since S^n is connected for $n \geq 1$, we have $b_0(S^n) = 1$.

Begin with the base case $n = 1$.

$$\begin{array}{cccc} & \underline{H^p(S^1)} & \underline{H^p(U) \oplus H^p(V)} & \underline{H^p(S^0)} \\ 0 \longrightarrow & \mathbb{R} \longrightarrow & \mathbb{R}^2 \longrightarrow & \mathbb{R}^2 \\ \hookrightarrow & \mathbb{R}^{b_1(S^1)} \longrightarrow & 0 \longrightarrow & 0 \end{array} \tag{13}$$

The ranks are determined sequentially by Claim 1:

$$\begin{array}{l} 0 \xrightarrow[0]{} \mathbb{R} \xrightarrow[1]{} \mathbb{R}^2 \xrightarrow[1]{} \mathbb{R} \\ \overset{1}{\hookrightarrow} \mathbb{R}^{b_1(S^1)} \xrightarrow[0]{} 0 \xrightarrow[0]{} 0 \end{array} \tag{14}$$

Thus $b_1(S^1) = 1$, so S^1 has the desired cohomology.

For $n > 1$, assume the claim holds for S^{n-1}. We will show it for S^n.

$$\begin{array}{cccc} & \underline{H^p(S^n)} & \underline{H^p(U) \oplus H^p(V)} & \underline{H^p(S^{n-1})} \\ 0 \longrightarrow & \mathbb{R} \longrightarrow & \mathbb{R}^2 \longrightarrow & \mathbb{R} \\ \hookrightarrow & \mathbb{R}^{b_1(S^n)} \longrightarrow & 0 \longrightarrow & 0 \\ \hookrightarrow & \mathbb{R}^{b_2(S^n)} \longrightarrow & 0 \longrightarrow & \cdots \\ & \cdots & 0 \longrightarrow & \mathbb{R} \\ \hookrightarrow & \mathbb{R}^{b_n(S^n)} \longrightarrow & 0 \longrightarrow & 0 \end{array} \tag{15}$$

Once again, the ranks are determined sequentially:

$$
\begin{array}{l}
0 \xrightarrow[0]{} \mathbb{R} \xrightarrow[1]{} \mathbb{R}^2 \xrightarrow[1]{} \mathbb{R} \\
\xrightarrow{0} \mathbb{R}^{b_1(S^n)} \xrightarrow[0]{} 0 \xrightarrow[0]{} 0 \\
\xrightarrow{0} \mathbb{R}^{b_2(S^n)} \xrightarrow[0]{} 0 \longrightarrow \cdots \\
\cdots \qquad 0 \xrightarrow[0]{} \mathbb{R} \\
\xrightarrow{1} \mathbb{R}^{b_n(S^n)} \xrightarrow[0]{} 0 \longrightarrow 0
\end{array}
\tag{16}
$$

Now the required Betti numbers are evident:

$$
b_p(S^n) = \begin{cases} 1 & \text{if } p = 0, n \\ 0 & \text{else,} \end{cases}
\tag{17}
$$

and the cohomology follows.

Random Note: What is the geometric meaning of $d^2 = 0$? If M is compact, oriented, $\eta \in \Omega^{n-2}(M)$, then

$$
0 = \int_M 0 = \int_M d^2\eta = \int_{\partial M} d\eta = \int_{\partial\partial M} \eta.
\tag{18}
$$

The statement $d^2 = 0$ is dual to the statement $\partial^2 = \varnothing$, i.e. the boundary of a boundary is empty. (If not, take η to be a bump function which is positive on $\partial\partial M$.)

Long exact sequences

Given a SES of chain complexes

$$
0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0,
\tag{19}
$$

we want to derive the LES of cohomology

$$\begin{array}{l} 0 \longrightarrow H^0(A) \xrightarrow{f^*} H^0(B) \xrightarrow{g^*} H^0(C) \xrightarrow{\partial^*} \\ \quad\longrightarrow H^1(A) \xrightarrow{f^*} H^1(B) \xrightarrow{g^*} H^1(C) \xrightarrow{\partial^*} \\ \quad\longrightarrow \cdots \end{array} \tag{20}$$

$$\begin{array}{l} \cdots \xrightarrow{\partial^*} \\ \quad\longrightarrow H^n(A) \xrightarrow{f^*} H^n(B) \xrightarrow{g^*} H^n(C) \longrightarrow \\ \quad\longrightarrow 0 \end{array}$$

Warning: The notation for f, g conflicts with our usual notation for de Rham cohomology. We had $\phi : M \to N$ would induce $\phi^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$, and $\phi^* : H^\bullet(N) \to H^\bullet(M)$. Above, the maps f, g correspond to maps cochain maps $\Omega^\bullet(N) \to \Omega^\bullet(M)$, while f^*, g^* correspond to cohomology maps $H^\bullet(N) \to H^\bullet(M)$.

We need to verify several things. First, let's verify exactness here:

$$H^p(A) \xrightarrow{f^*} H^p(B) \xrightarrow{g^*} H^p(C) \tag{21}$$

We must verify that if $g^*([b]) = 0$, then $[b] = f^*([a])$ for some $[a] \in H^p(A)$. We obtain this result via diagram chasing with

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^{p-1} & \xrightarrow{f} & B^{p-1} & \xrightarrow{g} & C^{p-1} & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d & & \\ 0 & \longrightarrow & A^{p} & \xrightarrow{f} & B^{p} & \xrightarrow{g} & C^{p} & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d & & \\ 0 & \longrightarrow & A^{p+1} & \xrightarrow{f} & B^{p+1} & \xrightarrow{g} & C^{p+1} & \longrightarrow & 0 \end{array} \tag{22}$$

where the rows are exact.

Proof. Begin with $[b] \in H^p(B)$ satisfying $g^*([b]) = 0$. Since $b \in B^p$ represents a cohomology class, $db = 0$. Because $0 = g^*([b]) = [g(b)]$, $g(b) = dc$ for some $c \in C^{p-1}$.

$$\begin{array}{ccc} & & c \\ & & \downarrow \\ b & \longmapsto & g(b) \\ \downarrow & & \\ 0 & & \end{array} \tag{23}$$

We want to conclude that $[b] = f^*([a])$, so we seek $a \in A^p$ with $da = 0$ and $b = f^*(a) + d\tilde{b}$ for some $\tilde{b} \in B^{p-1}$. Now g is surjective, so $c = g(\tilde{b})$:

$$\begin{array}{ccc} \tilde{b} & \longmapsto & c \\ & & \downarrow \\ b & \longmapsto & g(b) \\ \downarrow & & \\ 0 & & \end{array} \tag{24}$$

It's not necessarily true that $d\tilde{b} = b$. All we know is that $g(b) = d(g(\tilde{b})) = g(d\tilde{b})$. By linearity, $g(b - d\tilde{b}) = 0$. By exactness, $b - d\tilde{b} = f^*(a)$ for some $a \in A^p$.

$$\begin{array}{ccccc} a & \longmapsto & b - d\tilde{b} & \longmapsto & 0 \\ & & \downarrow & & \\ & & 0 & & \end{array} \tag{25}$$

and $d(b - d\tilde{b}) = db - dd\tilde{b} = 0 - 0$. Filling out,

$$\begin{array}{ccccc} a & \longmapsto & b - d\tilde{b} & \longmapsto & 0 \\ \downarrow & & \downarrow & & \\ da & \longmapsto & 0 & & \end{array} \tag{26}$$

Since f is injective, we know that $da = 0$. Thus we have $[a]$ with $f^*([a]) = [b]$. □

Next we define $\partial^* : H^p(C) \to H^{p+1}(A)$.

To define $\partial^*([c])$, we use the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^p & \xrightarrow{f} & B^p & \xrightarrow{g} & C^p & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d & & \\ 0 & \longrightarrow & A^{p+1} & \xrightarrow{f} & B^{p+1} & \xrightarrow{g} & C^{p+1} & \longrightarrow & 0 \end{array} \tag{27}$$

where we have

$$\begin{array}{ccccc} & & b & \longmapsto & c \\ & & \downarrow & & \downarrow \\ a & \longmapsto & db & \longmapsto & 0 \end{array} \tag{28}$$

and we define $\partial^*([c]) = [a]$.

It remains to verify that ∂^* is well-defined, and that both

$$H^p(B) \xrightarrow{g^*} H^p(C) \xrightarrow{\partial^*} H^{p+1}(A) \tag{29}$$

and

$$H^p(C) \xrightarrow{\partial^*} H^{p+1}(A) \xrightarrow{f^*} H^{p+1}(B) \tag{30}$$

are exact. All these are exercises in diagram chasing. Since diagram chasing becomes a mechanical procedure after a little practice, we leave these as an optional exercise.

Resolving the subspace of constant functions

If $U \subset M$ is a ball, then

$$0 \to \mathbb{R} \hookrightarrow \Omega^0(U) \to \Omega^1(U) \to \cdots \to \Omega^n(U) \to 0 \tag{31}$$

is exact. Thus

$$0 \to \Omega^0(U) \to \Omega^1(U) \to \cdots \to \Omega^n(U) \to 0 \tag{32}$$

resolves the subspace of constant functions $U \subset \mathbb{R}$. Any manifold is assembled from such balls, and Mayer-Vietoris allows one to compute cohomology based on how these balls fit together. Consequently, one can forget the de Rham complex, and compute cohomology abstractly, using Mayer-Vietoris.

The de Rham theorem states that if we use any other chain complex (satisfying certain technical conditions) which resolves the subspace of constant functions, then we always get something isomorphic to de Rham cohomology.

In the beginning, we were led to the de Rham complex by considering $\mathbb{R} = \ker \vec{\nabla}$. Now let's take an alternate route.

Suppose U is a ball. Define $\mathcal{C}^0(U)$ to be the vector space of arbitrary functions

$$\mathcal{C}^0(U) := \{f : U \to \mathbb{R}\}. \tag{33}$$

A function $f \in \mathcal{C}^0(U)$ is constant iff every continuous path $\gamma : [0,1] \to U$ satisfies $f(\gamma(1)) - f(\gamma(0)) = 0$.

Define $\mathcal{C}^1(U)$ to be the vector space of arbitrary functions

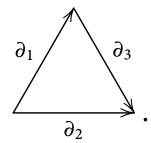
$$\mathcal{C}^1(U) := \{g : \{\text{paths in } U\} \to \mathbb{R}\} \tag{34}$$

and define the linear map

$$\begin{aligned} d : \mathcal{C}^0(U) &\to \mathcal{C}^1(U), \\ df &:= [\gamma \mapsto f(\gamma(1)) - f(\gamma(0))]. \end{aligned} \tag{35}$$

Then $f \in \mathcal{C}^0(U)$ is constant iff $df \equiv 0$. To proceed to the next level, we should resolve $\mathrm{im}(d) \subset \mathcal{C}^1(U)$.

Let Δ be a model triangle with oriented edges



(36)

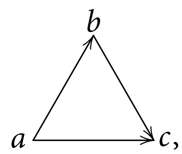
Define $\mathcal{C}^2(U)$ to be the vector space of arbitrary functions

$$\mathcal{C}^2(U) := \{h : \{\text{continuous maps } \sigma : \Delta \to U\} \to \mathbb{R}\} . \tag{37}$$

Define

$$\begin{aligned} d : \mathcal{C}^1(U) &\to \mathcal{C}^2(U), \\ dg &:= [\sigma \mapsto g(\sigma|_{\partial_1}) - g(\sigma|_{\partial_2}) + g(\sigma|_{\partial_3})] . \end{aligned} \tag{38}$$

Suppose Δ has vertices



(39)

and $g = df$. Then

$$\begin{aligned} d^2 f = dg &= [\sigma \mapsto g(\sigma|_{\partial_1}) - g(\sigma|_{\partial_2}) + g(\sigma|_{\partial_3})] \\ &= [\sigma \mapsto df(\sigma|_{\partial_1}) - df(\sigma|_{\partial_2}) + df(\sigma|_{\partial_3})] \\ &= \left[\sigma \mapsto \Big(f(\sigma(b)) - f(\sigma(a))\Big) - \Big(f(\sigma(c)) - f(\sigma(a))\Big) + \Big(f(\sigma(c)) - f(\sigma(b))\Big)\right] \\ &= [\sigma \mapsto 0] \\ &= 0. \end{aligned} \tag{40}$$

If U is a ball, then $dg = 0 \iff g = df$. Using simplices, we can continue the resolution, and obtain the complex of *singular cochains*

$$0 \to \mathcal{C}^0(U) \to \mathcal{C}^1(U) \to \mathcal{C}^2(U) \to \cdots \tag{41}$$

resolving $\mathbb{R} \subset \mathcal{C}^0(U)$.

For a manifold M which is not necessarily a ball, we form the complex

$$0 \to \mathcal{C}^0(M) \to \mathcal{C}^1(M) \to \mathcal{C}^2(M) \to \cdots \tag{42}$$

with singular cohomology

$$H^p_{\text{sing}}(M) \cong H^p_{dR}(M). \tag{43}$$

Singular cohomology makes sense for arbitrary topological spaces, not just manifolds.

Singular cohomology also makes sense with values any abelian group G in place of $\mathbb{R}$. Group-valued cohomology is much more powerful. For example, when $G = \mathbb{R}$,

$$G/(2G) = \mathbb{R}/(2\mathbb{R}) = \{0\} . \tag{44}$$

However, with $G = \mathbb{Z}$,

$$G/(2G) = \mathbb{Z}/(2\mathbb{Z}) = \mathbb{Z}_2. \tag{45}$$

We have come full-circle to algebraic topology, which relies neither on calculus nor manifolds.